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FOR THE

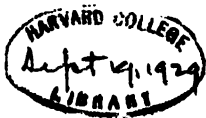
USE OF COLLEGES AND SCHOOLS.

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1831.

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ADVERTISEMENT.

THE following translation extends to the *General Theory of Equations*, and comprehends a little more than one half of the original work. In preparing it for the press considerable use has been made of a translation by Augustus De Morgan, Professor of Mathematics in the University of London, extending to about one fourth of the original treatise. Questions, intended as an exercise for the learner, are occasionally introduced. They were selected from the celebrated collection of Meier Heirsch.

Cambridge, July, 1881.



ERRATA.

Page	line	9, for	13 $a, b,$	read	13 $a^2 b^3.$
"	13,	" 10,	" $-b^3$	"	$+b^3.$
"	17,	" 18,	" $-20 b^2 a^2$	"	$-20 b^2 d^2.$
"	20,	" 22,	" $\times 3 a$	"	$\times (3 a.$
"	30,	" 5,	" A''	"	$A'.$
"	30,	" 23,	" $15 c^2$	"	$15 c^3.$
"	35,	" 23,	" $4 a^2$	"	$(4 a^2.$
"	40,	" 19,	" $12 b$	"	$-12 b.$
"	43,	" 15,	" $3 b - c$	"	$3 b + c.$
"	46,	" 12,	" $-b$	"	$+b.$
"	51,	" 18,	" $3 a b$	"	$a - 3 b.$
"	52,	" 3,	" factions	"	fractions.
"	57,	" 12,	" n_2	"	$n^2.$
"	74,	" 22,	" $+b$	"	$-b.$
"	91,	" 37,	" z into	"	z into.
"	94,	" 19,	" $b c' -$	"	$c b' -.$
"	97,	" 30,	" $d'' =$	"	$d' =.$
"	103,	" 6,	" more	"	greater.
"	104,	" 4,	" them	"	themselves.
"	109,	" 3,	" $3 a^3$	"	$3 a^2.$
"	119,	" 2,	" $\frac{27}{12}$	"	$\frac{27}{12}.$
"	"	" 27,	" $\frac{9}{2}$	"	$\frac{9}{2}.$
"	120,	" 16,	" $\frac{1}{12^2}$	"	$\frac{1}{(12)^2}.$
"	121,	" 15,	" 00,1	"	0,01.
"	135,	" 19,	" $(\sqrt{b} - \sqrt{c})$	"	$(\sqrt{b} + \sqrt{c}).$
"	138,	" 15,	" $a - \sqrt{c}$	"	$-a \sqrt{c}.$
"	157,	" 4,	" p^2	"	$n^2.$
"	158,	" 35,	" $a + x$	"	$(a + x).$
"	160,	" 23,	" $\sqrt{s^2}$	"	$\sqrt{s^2 - p}.$
"	161,	" 5,	" <i>unauxiliary known</i>	[ary. read <i>unknown auxili-</i>	
"	166,	" 3,	" formula	"	formula (1).
"	"	" 15,	" formula	"	formula (7).
"	"	" 17,	" $\sqrt{np + m^2} - m$	"	read $(\sqrt{np + m^2} - m).$
"	170,	" 25,	" $2 t''$	"	read $2 t'''.$
"	177,	" 14,	" 6 pieces	"	26 pieces.
"	182,	" 9,	" $a(a - b t$	"	$a(a - b t).$
"	183,	" 5,	dele à.		
"	"	" 6,	" à,		
"	184,	" 37, for	dy	"	read $y.$
"	185,	" 20,	300	"	3000.
"	"	" 30,	$\frac{1750}{73}$	"	$\frac{1750}{17}.$

Page 211, line 21, for the product read be the product.

" 215,	" 17,	" = 5	" + 5.
" 230,	" 19,	" $(3ab)^2$	" $(3ab)^2$.
" 233,	" 1,	" $\sqrt[n]{}$	" $(\sqrt[n]{})$.
" " 6,	" c^3	" c^2 .	" c^2 .
" 237,	" 27,	" $\sqrt[5]{5c}$	" $\sqrt[5]{5c}$.
" 239,	" last,	dele = 0.	
" 245,	" 22,	for $-\frac{3}{4}$	" $a^{-\frac{3}{4}}$
" 257,	" 12,	" $(q^n - 1$	" $(q^n - 1)$.

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ELEMENTS OF ALGEBRA.

INTRODUCTION.

1. **ALGEBRA** is the part of mathematics in which symbols are employed to abridge and generalize the reasonings made use of in the resolution of questions relating to numbers.

We distinguish questions into two leading classes, *theorems* and *problems*. A *theorem* has for its object to demonstrate the existence of certain properties of known and given numbers. In a *problem* it is proposed to determine certain numbers from other numbers that are known, and which have with the first relations indicated by the enunciation.

2. The principal elements used in algebra to arrive at this double object are ;

(1.) The letters of the alphabet, which serve to designate the numbers which form the subject of the reasoning. Their use is necessary, not only to abridge, but also to generalize the reasoning ; for by them it is more clearly seen that a property belongs at the same time to several numbers ; or, if the question be a problem, that the method of finding an answer to it is the same, whatever be the particular numbers stated in the question.

(2.) The sign $+$, which denotes the addition of two or more numbers, and is called *plus*. Thus $25 + 36$ is twenty-five plus thirty-six, or twenty-five added to thirty-six. Also $a + b$ is called *a plus b*, and is the number signified by *a*, augmented by the number signified by *b*.

(3.) The sign $-$, which is called *minus*, and denotes the subtraction of one number from another. Thus $45 - 24$ is called 45 minus 24, and is the difference between 45 and 24 ; $a - b$ is called *a minus b*, and is *a* diminished by *b*.

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(4.) The sign of multiplication, which is $\times$, or a point placed between the two quantities. Thus 36×25 , or $36 \cdot 25$, denotes 36 multiplied by 25, or the product of 36 by 25. When the numbers which are multiplied together are designated by letters, it is usual to write them one after the other without the interposition of any sign.

Thus ab means the same thing as $a \times b$ or $a \cdot b$; abc is the same as $a \times b \times c$, or $a \cdot b \cdot c$.

It must be understood that the notation ab or abc , which is more simple than $a \times b$ or $a \times b \times c$, can only be used when the numbers are designated by *letters*. For if we wished to represent the product of 5 multiplied by 6, and we wrote it for the sake of brevity 56, we should confound this product with the number fifty-six in the decimal system of notation.

This remark is very important to beginners.

(5.) The sign of division, $\div$ or $—$, the former being placed with the dividend preceding and the divisor following it, and the latter with the dividend above and the divisor below it.

Thus $24 \div 6$ or $\frac{24}{6}$ signifies 24 divided by 6, or the quotient of 24 divided by 6. $\frac{a}{b}$ means a divided by b , and is often expressed by saying a by b .

(6.) The *vinculum*, $—$ or $()$, the former being drawn over a quantity consisting of two or more letters, and the latter being made to enclose the compound quantity. Thus $(a + b) \times c$ signifies the product of the quantity $(a + b)$ by c , and not the product of b by c added to a , as we might suppose, if no such sign were used.

(7.) The *coefficient*, which is employed when a number, designated by a letter, is to be added several times to itself. Thus $a + a + a + a + a$, which represents the number a added four times to itself, is written $5a$. Also $11a$ represents a added to itself 10 times; $12ab$ represents the product ab added to itself 11 times.

The coefficient is the particular number written before a quantity, designated by one or several letters, which denotes the number of times plus one that this second quantity is added to itself.

(8.) The *exponent*, which is used when a number is multiplied several times by itself. Thus, instead of writing

$$a \times a \times a \times a \times a, \text{ or } a \cdot a \cdot a \cdot a \cdot a,$$

it is usual to write a^5 , which signifies a raised to the fifth power, or the product of a multiplied four times by itself. Thus b^5 is the same thing as $b \cdot b \cdot b \cdot b \cdot b$.

The exponent is a number written on the right of a letter and a little above it, and denotes how many times plus one the quantity, designated by the letter, is to be multiplied by itself, or how many times this letter is a factor in a product.

We give the name of *power* to the result of the multiplication of a number several times by itself, and the *degree* of the power, or the exponent, is the number of times which the letter is repeated in the expression of the power written at full length.

To show the importance of the exponent in algebra, suppose it were desired to express that a number a is multiplied three times by itself, that the product is then multiplied three times successively by b , and that this new product is multiplied twice successively by c . This is written simply $a^4 b^3 c^2$. If the last result is to be added six times to itself, or multiplied by 7, it is written thus, $7 a^4 b^3 c^2$. This gives an idea of the conciseness of algebraical language.

(9.) The sign $\sqrt{}$, which is placed before a number when the extraction of any root of that number is indicated. Thus $\sqrt[3]{a}$ is called the third or cube root of a ; $\sqrt[4]{b}$ is called the fourth root of b .

The second, third, &c. root of any number is the number which, when raised to the second, third, &c. power, reproduces that number.

(10.) The sign by means of which it is expressed that two quantities are equal to one another. This sign is $=$, and is read thus, *is equal to*, or *equals*.

Thus the way of expressing that the difference of 36 and 25 is 11, is $36 - 25 = 11$; that is, 36 minus 25 equals 11.

(11.) The sign of *inequality* $>$, which is used to express that one quantity is greater or less than another.

Thus $a > b$ signifies that a is greater than b , and $a < b$ signifies that a is less than b . The opening of the sign is always turned towards the greater quantity.

From what has been said we may evidently regard algebra as a species of language formed of a number of signs, by the aid of which the connexion of ideas is perceived with great facility in the reasoning which it is necessary to adopt, either to demonstrate the existence of a property, or to find the solution of a problem. A

better idea will be formed as to the utility of algebraical symbols from the following questions.

FIRST QUESTION.—PROBLEM.

3. The sum of two numbers is 67, their difference is 19; what are these two numbers?

We must establish, by the help of the signs, which have been defined, a connexion between the known and unknown numbers in the statement of the question.

If the less of the two numbers had been known, the greater might have been found by adding 19 to the other. Call the less of the two x , the greater is then $x + 19$, their sum is $x + x + 19$, or $2x + 19$. Now this sum is 67. We have therefore the equation $2x + 19 = 67$.

Now, if $2x$ augmented by 19 gives 67, $2x$ is less than 67 by 19, or $2x = 67 - 19 = 48$.

Therefore x is half of 48; that is, $x = \frac{48}{2} = 24$; and as the less number is 24, the greater, which is $x + 19$, is $24 + 19$, or 43. We may easily see that $43 + 24 = 67$, and that

$$43 - 24 = 19.$$

The algebraical operations contained in the above solution are merely the following. Let x be the less number. Then $x + 19$ is the greater. And, by the question, $2x + 19 = 67$.

Therefore $2x = 67 - 19 = 48$, and $x = \frac{48}{2} = 24$.

Consequently $x + 19 = 24 + 19 = 43$.

Another Solution.

Let x be the greater number.

Then $x - 19$ is the less, and $2x - 19 = 67$.

Therefore $2x = 67 + 19 = 86$, and $x = \frac{86}{2} = 43$.

$x - 19$ is therefore $43 - 19$, or 24.

The student may now see how all the reasonings necessary for the solution of a problem may be contained in a very small space; though in this case, had they been written in ordinary language, they would have occupied more than a page.

General Solution of this Problem.

4. The sum of two numbers is a , and their difference b . What are the two numbers?

Let x be the less number.

Then $x + b$ is the greater,

and

$$2x + b = a.$$

Therefore

$$2x = a - b, \text{ and } x = \frac{a - b}{2} = \frac{a}{2} - \frac{b}{2}.$$

Consequently

$$x + b = \frac{a}{2} - \frac{b}{2} + b = \frac{a}{2} + \frac{b}{2}.$$

As the form of these results is the same, whatever value be given to a and b , it follows that *the greater of two numbers is equal to half their sum plus half their difference, and that the less is equal to half their sum minus half their difference.*

Thus, let the sum of two numbers be 237, and the difference 99; the greater will be

$$\frac{237}{2} + \frac{99}{2} \text{ or } \frac{237 + 99}{2} \text{ or } \frac{336}{2} = 168;$$

and the less will be

$$\frac{237}{2} - \frac{99}{2} \text{ or } \frac{138}{2} = 69.$$

Indeed

$$168 + 69 = 237, \text{ and } 168 - 69 = 99.$$

From this solution is seen the utility of representing by letters the given quantities in a problem. As the arithmetical operations upon the letters can only be indicated, the result at which we arrive retains indications of the several operations which must be performed on the known quantities in order to find the unknown ones.

The expressions

$$\frac{a}{2} + \frac{b}{2} \text{ and } \frac{a}{2} - \frac{b}{2},$$

obtained in the preceding problem, are called in algebra *formulas*, because they may be regarded as containing the solutions of all questions of the same nature, in the enunciation of which the only things which vary are the numerical values of the given quantities.

SECOND QUESTION.—THEOREM.

5. *The sum of two numbers multiplied by their difference is equal to the difference of the squares, or second powers of those numbers.*

Let 12 and 9 be the two numbers; their sum is 21, and their difference 3; and the product 21×3 , or 63, is equal to 144, the square of 12, diminished by 81, the square of 9.

But to make this property known, whatever be the two numbers, let us represent them by a and b .

Their sum will be $a + b$, and their difference $a - b$. To form the product of these two expressions, first multiply the sum $a + b$ by a . This product will be $a \times a + b \times a$, or, more simply, $a^2 + ab$; for we must take each of the two parts of which $a + b$ is composed, as many times as there are units in a , and add together the two products. But it is not by the whole of a that we are to multiply, but by a diminished by b ; therefore the product $a^2 + ab$ is too great by $(a + b) \times b$; that is, by $ab + b^2$. We must then subtract $ab + b^2$ from $a^2 + ab$, the algebraical representation of which is $a^2 + ab - ab - b^2$; and as $+ab$, $-ab$ cancel each other, the product required is $a^2 - b^2$.

This result being the same, whatever values are given to a and b , it follows that the theorem is true for any numbers whatever.

THIRD QUESTION.—THEOREM.

6. *If to the two terms of a proper fraction, that is, of a number less than unity, the same number be added, the resulting fraction is greater than the first.*

Let $\frac{1}{2}$ be the proposed fraction; add 3 to each of its terms, and it becomes $\frac{4}{5}$. These two fractions, when reduced to a common denominator, become $\frac{7}{10}$ and $\frac{5}{10}$. Now the second fraction is evidently greater than the first.

To ascertain whether this theorem is true, whatever be the proposed fraction, let us designate this fraction by $\frac{a}{b}$, supposing $a < b$. Let m be the number to be added to both terms of the fraction; we shall have

$$\frac{a+m}{b+m}.$$

To compare these two fractions, it is necessary to reduce them to a common denominator. For this purpose, multiply the two terms of the first fraction by $b + m$, and of the second by b . Now, to multiply a by $b + m$ is to take a as many times as there are units in b , and again as many times as there are units in m , which gives $ab + am$. Also, the product of b by $b + m$ is $b^2 + bm$, which gives $\frac{ab + am}{b^2 + bm}$ for the first fraction. In the same way, if we multiply the two terms of the second fraction $\frac{a + m}{b + m}$ by b , it becomes $\frac{ab + bm}{b^2 + bm}$.

The two numerators $ab + am$, $ab + bm$, have a common part ab , and the part bm of the second numerator is greater than the part am of the first, because $b > a$. Therefore the second fraction is greater than the first, which was to be proved.

We may see also by the foregoing reasoning that $\frac{a}{b}$ must be a proper fraction, in order that the theorem may be true; for otherwise the contrary of this theorem would be true, since we should then have $ab + bm < ab + am$.

7. By reflecting upon the solutions which have been given of the preceding question, it will be perceived that the employment of algebraical signs gives rise to rules common to several questions. Thus, for instance, in the second and third questions, we have been led to perform the multiplication of a sum $a + b$ by a number a , of a sum $a + m$ by b , and of a number a by a sum $b + m$. It appears then, that by establishing general rules for finding the results of all the operations which we have occasion to perform on algebraical quantities, we arrive at settled methods for resolving all questions relative to numbers by means of algebraical symbols.

This part of algebra may be entitled, the method of performing all the operations of arithmetic on algebraical or literal quantities; that is on numbers represented by algebraical symbols.

Of Algebraical Operations.

8. EVERY quantity written in algebraical language, is called an *algebraic quantity*, or the *algebraic expression for the proposed quantity*.

Thus $3a$ is the algebraic expression for three times the number a , $5a^2$ is the algebraic expression for five times the square of a , $2a^2 - 3ab + 4b^2$ is the expression for twice the square a , diminished by three times the product of a and b , and then augmented by four times the square of b .

We give the name of *monomial* or *simple quantity* to a quantity consisting of one term only, or which is not joined to any other by the sign of addition or subtraction; and that of *polynomial* to a quantity consisting of more terms than one, or an expression composed of several parts separated from one another by the signs $+$ or $-$.

Thus $3a$, $5a^2$, $7a^3b^2$, are monomials; $3a - 5b$, $2a^2 - 3ab + 4b^2$ are polynomials. The first of these two expressions is called a *binomial*, because it is composed of two terms. The second is called a *trinomial*, because it is composed of three terms.

9. The *numerical value* of an algebraical expression is the number which would be obtained, if particular values were given to the letters which it contains, and all the arithmetical operations indicated, were performed. This numerical value evidently depends on the particular values which are given to the letters, and generally varies with them. Thus $2a^3$ has for its numerical value 54, when a is made equal to 3, for the cube of 3 is 27, and twice 27 is 54. The numerical value of the same expression is 250 when a is made equal to 5, for the cube of 5 is 125, and twice 125 is 250.

We have said that the numerical value of the expression will generally vary with those of the letters composing it; in some cases, the numerical value of an algebraical expression may remain constant, although the values of the letters which compose it vary. Thus, in the expression $a - b$, as long as values are given to a and b , which increase, by equal degrees, the expression will not change.

For example, let $a = 7$, $b = 4$, then $a - b = 3$.

Let $a = 7 + 5$, or 12, and $b = 4 + 5$, or 9, still $a - b = 3$, &c.

The numerical value of a polynomial does not change when the order of the terms is altered, provided care be taken to preserve to each term its proper sign. Thus the polynomials $4a^3 - 3a^2b + 5ac^2$, $5ac^2 - 3a^2b + 4a^3$, and $4a^3 + 5ac^2 - 3a^2b$, have

the same numerical value. This is an evident consequence of the nature of arithmetical addition and subtraction. This observation will be very useful in what follows.

10. Of the different terms which compose a given polynomial, those which are preceded by the sign $+$ are called *positive* terms, and those which are preceded by the sign $-$, *negative* terms. These denominations are improper, but custom has sanctioned them.

The first term of a polynomial is not usually preceded by any sign, and then it is supposed to have the sign $+$.

11. Each of the literal factors which compose a term, is called a *dimension* of that term; and the *degree* of a term is the number of these factors or dimensions. The coefficient is not counted as a dimension. Thus $3a$ is a term of one dimension, or of the first degree; $5ab$ is a term of two dimensions, or of the second degree; $7a^3bc^2$, being the same thing as $7aaaa bcc$, is of six dimensions, or of the sixth degree.

In general, the degree, or number of dimensions of a term, is the sum of the exponents of the letters which compose it. It may be remarked also, that from the definition of an exponent, a letter, which has no exponent, is considered as having unity for its exponent. Thus the degree of the term $8a^2bcd^3$ is

$$2 + 1 + 1 + 3, \text{ or } 7.$$

A polynomial is called *homogeneous* when its terms are all of the same degree.

$3a - 2b + c, 3a^3 - 4ab + b^2, 5a^2c - 4c^3 + 2c^2d,$
are all homogeneous;

$$8a^3 - 4ab + c,$$

is not homogeneous.

12. *Similar* terms are those which are composed of the same letters with the same exponents. Thus $7ab, 3ab$, are similar terms, and $4a^3b^2, 5a^3b^2$, are also similar; $8a^2b$ and $7a^2b$ are not similar terms; for though they are composed of the same letters, each letter has not the same exponent in both. It often happens that a polynomial contains several similar terms in its expression, and in that case it admits of simplification.

Let the polynomial be

$$4a^3b - 3a^2c + 9ac^2 - 2a^2b + 7a^2c - 6b^3.$$

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This can (9) be written thus,

$$4a^2b - 2a^2b + 7a^2c - 3a^2c + 9ac^2 - 6b^3.$$

Now $4a^2b - 2a^2b$ is evidently $2a^2b$; $7a^2c - 3a^2c$ is $4a^2c$.

Then the polynomial itself becomes

$$2a^2b + 4a^2c + 9ac^2 - 6b^3.$$

Let there be in any polynomial the terms

$$+ 2a^3bc^2, - 4a^3bc^2, + 6a^3bc^2, - 8a^3bc^2, + 11a^3bc^2.$$

First, the sum of the positive terms

$$+ 2a^3bc^2 + 6a^3bc^2 + 11a^3bc^2,$$

is equal to

$$+ 19a^3bc^2;$$

the sum of the negative terms

$$- 4a^3bc^2 - 8a^3bc^2, \text{ is } - 12a^3bc^2.$$

The result of the five terms is therefore

$$19a^3bc^2 - 12a^3bc^2, \text{ or } 7a^3bc^2.$$

It may happen that the sum of the negative terms is greater than that of the positive. In this case, subtract the positive coefficient from the negative one, and place the sign — before the result. Thus, if $5a^2b$ be the sum of the positive terms, and $-8a^2b$ of the negative ones; since $-8a^2b$ is equivalent to

$$- 5a^2b - 3a^2b, \text{ then } 5a^2b - 8a^2b$$

is equivalent to

$$5a^2b - 5a^2b - 3a^2b, \text{ or to } - 3a^2b.$$

Whence this rule may be deduced;—To reduce several similar terms to one, form one positive term of all the positive similar terms, which is done by adding the coefficients of these terms, and affixing to their sum the literal part common to them all. Form, by the same means, one single negative term of all the terms which have the sign —; subtract the less sum from the greater, and give to the result the sign of the greater.

It is important to observe, that the reduction affects only the coefficients, and never the exponents.

We shall find, from this rule, that

$$6a^2b - 8a^2b - 9a^2b + 15a^2b - a^2b = + 3a^2b$$

$$7abc^2 - abc^2 - 7abc^2 - 8abc^2 + 4abc^2 = - 5abc^2.$$

The reduction of similar terms is a species of operation peculiar to algebra, which appears in algebraical addition, subtraction, mul-

multiplication, and division. These operations we now proceed to develop.

Of Algebraic Addition.

13. If the quantities $3a$, $5b$, $2c$ be added, the result of this addition is denoted thus, $3a + 5b + 2c$; an expression which does not admit of being simplified.

The result of the addition of the simple quantities $4a^2b^3$, $2a^2b^3$, and $7a^2b^3$, is $4a^2b^3 + 2a^2b^3 + 7a^2b^3$, which when reduced becomes $13a^2b^3$.

Take the polynomials

$$3a^2 - 4ab, 2a^2 - 3ab + b^2, 2ab - 5b^2.$$

To form one polynomial which will express the sum of these, observe, that to add to the number expressed by $3a^2 - 4ab$, that expressed by $2a^2 - 3ab + b^2$, is the same thing as adding the difference between the number of units in $2a^2 + b^2$, and the number of units in $3ab$, which could easily be done if particular values were given to a and b . But as this cannot be done in the actual state of the quantities, it will amount to the same thing to add $2a^2 + b^2$ to $3a^2 - 4ab$, and then to subtract $3ab$, which gives

$$3a^2 - 4ab + 2a^2 + b^2 - 3ab;$$

or, changing the order of the terms (9),

$$3a^2 - 4ab + 2a^2 - 3ab + b^2.$$

In like manner, to add $2ab - 5b^2$ to this last expression, it is sufficient to write

$$3a^2 - 4ab + 2a^2 - 3ab + b^2 + 2ab - 5b^2.$$

It now remains to reduce the similar terms (12), and the result is

$$5a^2 - 5ab - 4b^2.$$

Since analogous reasonings may be applied to all polynomials, we may deduce this general rule for the addition of two or more polynomials:—*Write the given polynomials one after the other, preserving to the terms which compose them their respective signs, and reduce the similar terms if there are any.*

The following are examples; it is required to add

$$I. 3a^2 - 4ab - 2b^2$$

$$5a^2 + 2ab - b^2$$

$$3ab - 2b^2 - 3c^2$$

The result is,

$$8a^2 + ab - 5b^2 - 3c^2.$$

$$\begin{aligned} \text{II. } 7a^2b - 3abc - 8b^2c - 9c^3 + cd^2 \\ 8abc - 5a^2b + 3c^3 - 4b^2c + cd^2 \\ 4a^2b - 8c^3 + 9b^2c - 3d^3. \end{aligned}$$

The result of the addition is,

$$6a^2b + 5abc - 3b^2c - 14c^3 + 2cd^2 - 3d^3.$$

In practice, the given quantities are usually placed under each other, as in these two examples; the reduction of the similar terms is then made, and the results written down with their respective signs.

Thus, in the first example, as the term $3a^2$ in the first line is similar to the term $5a^2$ in the second, write $8a^2$ as the result of the reduction of these terms. Pass then to the term $-4ab$, and reduce it with the terms $+2ab$ and $3ab$, which gives $+ab$. Write this by the side of $8a^2$. Continue the operation thus until all the terms have been reduced.

Of Algebraic Subtraction.

14. The result of the subtraction of $4b$ from $5a$, is $5a - 4b$.

In like manner the difference between $7a^2b$ and $4a^2b$ is $7a^2b - 4a^2b$, or $3a^2b$.

Let it now be required to subtract $2b - 3c$ from $4a$. The result may be expressed in this form, $4a - (2b - 3c)$, by putting the quantity to be subtracted between parentheses, and writing it after the first quantity with the sign $-$. But it is often necessary to form a single polynomial from this expression; and in this consists, for the most part, the rule for algebraic subtraction.

Accordingly, observe, that if a , b , and c , were given numerically, and the operations indicated by $2b - 3c$ performed, the result must be subtracted from $4a$. But as this cannot be done in the actual state of the quantities, we begin by subtracting $2b$ from $4a$, which gives $4a - 2b$. But in taking away $2b$ units, we have taken away a quantity too great by $3c$ units; the result must then be rectified by adding to it $3c$. Thus $4a - 2b + 3c$ is the result of the subtraction required.

Let it be required to subtract $5a^2 - 4ab + 3bc - b^2$ from $8a^2 - 2ab$; the operation may be indicated by the following expression,

$$8a^2 - 2ab - (5a^2 - 4ab + 3bc - b^2).$$

To reduce this expression to a single polynomial, observe, that to

subtract $5a^2 - 4ab + 3bc - b^2$ is equivalent to subtracting the difference between the sum $5a^2 + 3bc$ of the positive terms, and the sum $4ab + b^2$ of the negative ones. We first take away $5a^2 + 3bc$, which gives $8a^2 - 2ab - 5a^2 - 3bc$; but as this result is necessarily too little by $4ab + b^2$, we must add this last quantity; it thus becomes

$$8a^2 - 2ab - 5a^2 - 3bc + 4ab + b^2,$$

or $8a^2 - 2ab - 5a^2 + 4ab - 3bc + b^2,$

if the terms are placed in their order. Reduced this becomes

$$3a^2 + 2ab - 3bc - b^2.$$

From what has been said, the following rule may be deduced;

To subtract one polynomial from another, write the polynomial to be subtracted after the other, changing the sign of each of its terms, and reduce the similar terms of the resulting expression, if there are any. Examples;

$$\text{I. } \left. \begin{array}{r} 5a^3 - 4a^2b + 3b^2c \\ -(3ab - 2a^2 - 8b^2c) \end{array} \right\} = 7a^3 - 7a^2b + 11b^2c.$$

$$\text{II. } \left. \begin{array}{r} 4ab - cd - 2b^2 + 3a^2 \\ -(5ab - 4cd + 3b^2 + 3a^2) \end{array} \right\} = -ab + 3cd - 5b^2.$$

15. By this rule, certain transformations may be made in polynomials.

For example,

$$6a^2 - 3ab + 2b^2 - 2bc$$

is reduced to

$$6a^2 - (3ab - 2b^2 + 2bc);$$

and, in like manner,

$$7a^3 - 8a^2b - 4b^2c + 6b^3$$

is reduced to

$$7a^3 - (8a^2b + 4b^2c - 6b^3),$$

or

$$7a^3 - 8a^2b - (4b^2c - 6b^3).$$

These transformations, which consist in decomposing a polynomial into two parts separated from one another by the sign $-$, are very useful in algebra.

Of Algebraic Multiplication.

16. We take for granted a principle which is generally admitted in books on arithmetic, that the product of two or more numbers is the same, whatever be the order in which they are multiplied. (*Lacroix's Arith.* 26.)

Let us consider first the case where a simple quantity is to be multiplied by another simple quantity; let it be required to multiply $7a^3b^2$ by $4a^2b$. The expression of the product is written thus, $7a^3b^2 \times 4a^2b$. But this may be simplified, since by the preceding principle, and the signification of algebraic symbols (2), it is the same as $7 \times 4 \times a a a a b b b$. Now, as the coefficients are particular numbers, they may be formed into one by multiplying them together, which gives 28 for the coefficient of the product.

The product $a a a a a$ is a^5 , and the product $b b b$ is b^3 ; so that the final result is $28a^5b^3$.

Let $12a^2b^4c^2$ be multiplied by $8a^3b^2d^2$, the product is,

$$12 \times 8 \times a a a a a b b b b b c c d d, \text{ or } 96a^5b^6c^2d^2.$$

From these we may see that, to multiply two simple quantities by one another, we must,

- (1.) *Multiply the two coefficients together;*
- (2.) *Write after this product all the letters which enter into the multiplicand and multiplier, giving to each letter an exponent equal to the sum of the corresponding exponents in the two factors;*
- (3.) *If a letter enter into one only of the factors, write it in the product with the exponent which it has in that factor. The rule relative to the coefficients is attended with no difficulty.*

The reason for the rule respecting the exponents is, that any number a ought to be as many times a factor in the product as it is in both the multiplier and the multiplicand. Now the exponent of a letter equals (2) the number of times which it enters as a factor; therefore the sum of the two exponents of the same letter marks the number of times which it ought to be a factor in the product required. We shall find from the preceding rule that

$$\begin{aligned} 8a^2bc^2 \times 7abcd^2 &= 56a^3b^2c^3d^2, \\ 21a^3b^2cd \times 8abc^3 &= 168a^4b^3c^4d, \\ 4abc \times 7df &= 28abcd f. \end{aligned}$$

17. We now come to the multiplication of polynomials.

Let there be two polynomials, $a + b + c$, and $d + f$, composed of terms which are all positive or additive, the product may be expressed in this form, $(a + b + c)(d + f)$. But it is often necessary to form a single polynomial of the product thus indicated, and in this consists the rule for the multiplication of algebraical quantities.

It is evident that the multiplication of $a + b + c$ by $d + f$ is the same thing as taking $a + b + c$ as many times as there are units in d , then as many times as there are units in f , and adding together the two products. But to multiply $a + b + c$ by d , is to take each of the parts of the multiplicand d times, and to add these partial products, which gives $ad + bd + cd$. In like manner, to multiply $a + b + c$ by f , is to take each part of the multiplicand f times, and add the partial products. In this way we find,

$$(a + b + c)(d + f) = ad + bd + cd + af + bf + cf.$$

Thus to multiply two polynomials, composed of positive terms, multiply successively each of the terms of the multiplicand by each of the terms of the multiplier, and add all the products.

If the terms have coefficients or exponents, follow the rules laid down (16) for the multiplication of simple quantities. For example, $(3a^2 + 4ab + b^2)(2a + 5b)$ gives for its product

$$6a^3 + 8a^2b + 2ab^2 + 15a^2b + 20ab^2 + 5b^3,$$

or by reducing

$$6a^3 + 23a^2b + 22ab^2 + 5b^3.$$

To take a more general case, we must begin by observing, that if the multiplicand contains negative as well as positive terms this factor expresses the difference between the number of units in the sum of its positive terms and the sum of its negative ones. And the same holds true of the multiplier. It follows that the multiplication of any two polynomials is reduced to the multiplication of two binomials, such as $a - b$, $c - d$, where a denotes the sum of the positive terms, and $-b$ of the negative terms of the multiplicand, and c and $-d$ stand for the same things in the multiplier. We can now effect the multiplication expressed by $(a - b)(c - d)$. To multiply $a - b$ by $c - d$ is evidently to take $a - b$ as many times as there are units in c , and to subtract from it $a - b$, taken as many times as there are units in d , in other words, we are to multiply $a - b$ by c , and to subtract $a - b$, multiplied by d . But

$$(a - b) \times c = ac - bc, \text{ and } (a - b) \times d = ad - bd;$$

and since the second product is to be subtracted from the first, we must (14) change the signs of $ad - bd$, and write them after $ac - bc$, which gives

$$(a - b)(c - d) = ac - bc - ad + bd.$$

The least reflection on the manner in which this product has been formed, will show that, in all cases of multiplication, each positive term of the multiplier must be multiplied by each term of the multiplicand, and each partial product must have the same sign as the corresponding term of the multiplicand; that each negative term of the multiplier must be multiplied by each of the terms of the multiplicand, and each partial product must have the opposite sign to that of the corresponding term in the multiplicand. In the partial multiplications of each term of the multiplier by each term of the multiplicand, the rules laid down in article 16 must be observed.

For example, multiply

$$\begin{array}{r} 4a^3 - 5a^2b - 8ab^2 + 2b^3 \\ \text{by } 2a^2 - 3ab - 4b^2 \\ \hline 8a^5 - 10a^4b - 16a^3b^2 + 4a^2b^3 \\ - 12a^4b + 15a^3b^2 + 24a^2b^3 - 6ab^4 \\ - 16a^3b^2 + 20a^2b^3 + 32ab^4 - 8b^5 \\ \hline 8a^5 - 22a^4b - 17a^3b^2 + 48a^2b^3 + 26ab^4 - 8b^5. \end{array}$$

After having placed the polynomials under one another, multiply each of the terms of the first by the term $2a^2$ of the second, which gives

$$8a^5 - 10a^4b - 16a^3b^2 + 4a^2b^3,$$

whose signs are the same as those of the multiplicand. Pass then to the second term $3ab$ of the multiplier; and as this has a negative sign, in multiplying all the terms of the multiplicand by it, take care to give each product the opposite sign to that of its corresponding term in the multiplicand, and we shall have

$$- 12a^4b + 15a^3b^2 + 24a^2b^3 - 6ab^4.$$

Write this under the first.

The same operation, with the term $4b^2$, which is negative, will give

$$- 16a^3b^2 + 20a^2b^3 + 32ab^4 - 8b^5.$$

By reducing the similar terms we shall have for the product,

$$8a^5 - 22a^4b - 17a^3b^2 + 48a^2b^3 + 26ab^4 - 8b^5.$$

The rule for the signs which it is most important to remember in the multiplication of polynomials, is the following. *Whenever two terms of the multiplier and multiplicand have the same sign, the corresponding product has the sign +; and when they have different signs the product has the sign —.*

This is expressed in algebraical language by saying that + multiplied by +, or — multiplied by —, gives +; and that — multiplied by +, or + multiplied by —, gives —. But this rule, which is unintelligible by itself (because it is not known what is meant by the multiplication of two symbols, not of quantities, but of arithmetical operations), ought only to be regarded as an abbreviation of the preceding one. This is not the only case in which algebraists, to save words, have employed expressions which are incorrect, but which have the advantage of fixing rules in the memory.

The following are examples of the rule;

$$\begin{array}{l} \text{I.} \quad 3a^2 - 5bd + cf \\ \quad - 5a^2 + 4bd - 8cf \end{array}$$

$$- 15a^4 + 37a^2bd - 29a^2cf - 20b^2a^2 + 44bcdcf - 8c^2f^2$$

$$\begin{array}{l} \text{II.} \quad 4a^3b^2 - 5a^2b^3c + 8a^2bc^2 - 3a^2c^3 - 7abc^3 \\ \quad 2ab^2 - 4abc - 2bc^2 + c^3 \end{array}$$

$$\begin{array}{l} 8a^4b^4 - 10a^3b^4c + 28a^3b^3c^2 - 34a^3b^2c^3 \\ - 4a^2b^3c^3 - 16a^4b^3c + 12a^3bc^4 + 7a^2b^2c^4 \\ + 14a^2bc^5 + 14ab^2c^5 - 3a^2c^5 - 7abc^5. \end{array}$$

18. There are several important remarks to be made on algebraic multiplication.

First. If the polynomials, whose product is required, are homogeneous (11) (and the greater part of the questions in which the aid of algebra is wanted, particularly those of geometry, lead to similar expressions), *the product of the two is also homogeneous*; this is an evident consequence of the rules relative to letters and exponents in the multiplication of simple quantities. Moreover, *the degree of each-term of the product will be equal to the sum of the degrees of any two terms of the multiplicand and multiplier*. Thus, in the first of the two preceding examples, all the terms both of the multiplicand and multiplier being of the second degree, all the terms of the product are of the fourth degree. In the second, the multiplicand being of the fifth degree, and the multiplier of the

third, the product is of the eighth degree. This remark serves in practice to discover any errors of calculation in the exponents. For example, if in one of the terms of a product which ought to be homogeneous, the sum of the exponents is equal to 6, while in all the rest it is 7, there is an error in the addition of the exponents, and then we must go back to the multiplication of the two terms which produced that partial product.

Secondly. When there is no reduction of similar terms in the product, *the whole number of terms in the product is equal to the product of the number of terms in the multiplicand multiplied by the number of terms in the multiplier.* This is a consequence of the rule (17).

Thus, when there are five terms in the multiplicand, and four in the multiplier, there are 5×4 or 20 in the product. In general, if the multiplicand is composed of m terms, and the multiplier of n , the product will contain $m \times n$ terms.

Thirdly. When there are similar terms, the whole number of terms in the reduced product may be much less. But among the different terms of the product, there are those which cannot be reduced with any others. These are,

(1.) *That which arises from the multiplication of the term of the multiplicand, which has the highest exponent of one of the letters, by that term of the multiplier which has the highest exponent of the same letter.*

(2.) *That which arises from the multiplication of the two terms which have the least exponent of the same letter.*

These two products contain that letter with a higher and lower exponent than can be found in any other term, consequently they cannot be similar to any other term. This remark, the truth of which may be seen from the rule of the exponents, will be of great use in division.

19. To finish the subject of algebraical multiplication, we shall subjoin some results of frequent use in algebra.

(1.) The square, or second power of the binomial $a + b$. On the preceding principles,

$$(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2;$$

that is, *the square of the sum of two quantities is composed of the square of the first added to the square of the second, added to twice the product of the first by the second.*

The square of $5a^2 + 8a^2b$ will be found to be,

$$25a^4 + 80a^3b + 64a^4b^2.$$

(2.) The square of $a - b$.

$$(a - b)^2 = (a - b)(a - b) = a^2 - 2ab + b^2;$$

that is, *the square of the difference of two quantities is composed of the square of the first added to the square of the second, minus twice the product of the first and second.* Thus,

$$(7a^2b^2 - 12ab^2)^2 = 49a^4b^4 - 168a^3b^5 + 144a^2b^6.$$

(3.) The product of $a + b$ and $a - b$.

$$(a + b)(a - b) = a^2 - b^2;$$

that is, *the sum of two quantities multiplied by their difference, gives the difference of their squares.* This is the theorem demonstrated in article 5. Thus,

$$(8a^3 + 7ab^2)(8a^3 - 7ab^2) = 64a^6 - 49a^2b^4.$$

By combining these results the products of certain polynomials may be found more readily than by the ordinary methods. For example, to multiply

$$5a^2 - 4ab + 3b^2 \text{ by } 5a^2 - 4ab - 3b^2.$$

The first of these two quantities is the sum of the two numbers $5a^2 - 4ab$ and $3b^2$, and the second is their difference. Their product is therefore

$$(5a^2 - 4ab)^2 - (3b^2)^2, \text{ or } 25a^4 - 40a^3b + 16a^2b^2 - 9b^4.$$

20. It will be observed that in the results of multiplication which have been obtained, their composition, or the method of obtaining them from the multiplicand and multiplier, is independent of the particular values of a and b , the letters which enter into the factors.

The manner in which an algebraical product is formed from its two factors is called the LAW of that product, and this law is always the same, whatever be the values of the letters which enter into the factors.

21. The factors of a given polynomial are sometimes evident on inspection, and the decomposition is often useful.

Let the polynomial be $25a^4 - 30a^3b + 15a^2b^2$, it is evident that the factors 5 and a^2 enter into each of its terms. Thus, the polynomial can be put under the form

$$5a^2(5a^2 - 6ab + 3b^2).$$

In like manner, $64 a^4 b^4 - 25 a^2 b^6$ may be transformed into

$$(8 a^2 b^2 + 5 a b^4) (8 a^2 b^2 - 5 a b^4).$$

For, as $64 a^4 b^4$ and $25 a^2 b^6$ are the squares of $8 a^2 b^2$ and $5 a b^4$, it follows that the expression is the difference of two squares, and (19) that it is the sum of the roots of these squares multiplied by their difference.

Examples in Multiplication of Compound Quantities.

1. $(3 a + 3 b + 4 c) \times (3 a + 3 b - 4 c)$
 $= 9 a^2 + 18 a b + 9 b^2 - 16 c^2.$
2. $(4 a + 4 b - 3 c - 6 d) \times (4 a + 4 b + 3 c + 6 d)$
 $= 16 a^2 + 32 a b + 16 b^2 - 9 c^2 - 36 c d - 36 d^2.$
3. $(5 a^2 - 3 a b + 7 b^2) \times (3 a - b)$
 $= 15 a^3 - 14 a^2 b + 24 a b^2 - 7 b^3.$
4. $(5 a b + 3 a c - 4 b c) \times (7 a b - 18 a c + 2 b c + d)$
 $= 35 a^2 b^2 - 69 a^2 b c - 18 a b^2 c + 5 a b d - 54 a^2 c^2$
 $+ 78 a b c^2 + 3 a c d - 8 b^2 c^2 - 4 b c d.$
5. $(a + b + c + d) \times (a - b - c - d)$
 $= a^2 - b^2 - 2 b c - 2 b d - c^2 - 2 c d - d^2.$
6. $(-2 a + 3 b - c^2) \times (-3 f - 7 a + c^2)$
 $= 6 a f - 9 b f + 3 c^2 f + 14 a^2 - 21 a b + 5 a c^2$
 $+ 3 b c^2 - c^4.$
7. $(3 a + 4 c - 5 d) + (6 b - 7 n - 6 m) \times 3 a + 4 c$
 $- 5 d - (6 b - 7 n - 6 m)$
 $= 9 a^2 + 24 a c - 30 a d + 16 c^2 - 40 c d + 25 d^2$
 $- 36 b^2 + 84 b n - 49 n^2 + 72 m b - 84 n m - 36 m^2.$

Division.

22. Algebraical and arithmetical division have the same object, viz. when the product and one of the factors is given, to find the other factor.

Let us consider first the case of two simple quantities.

To divide $72 a^5$ by $8 a^3$ the result of which is expressed thus $\frac{72 a^5}{8 a^3}$. A third quantity is wanted, which, when multiplied by the second, will produce the first. Now, from the rules for the multiplication of simple quantities, the quantity sought is such that its

coefficient multiplied by 8 gives 72, and the exponent of a added to 3, its exponent in the divisor, gives 5, the exponent of the dividend. This quantity is therefore found by dividing 72 by 8, and subtracting the exponent 3 from the exponent 5, which gives

$$\frac{72 a^5}{8 a^3} = 9 a^2,$$

and it is plain that $8 a^3 \times 9 a^2 = 72 a^5$.

By the same rule

$$\frac{35 a^3 b^3 c}{7 a b} = 5 a^{3-1} b^{3-1} c = 5 a^2 b c;$$

and

$$7 a b \times 5 a^2 b c = 35 a^3 b^2 c.$$

Hence it will be seen that to divide one simple quantity by another, it is necessary,

- (1.) *To divide the coefficient of one by that of the other ;*
- (2.) *To write after this quotient the letters which are common to the dividend and divisor, giving each an exponent equal to the excess of its exponent in the dividend above that in the divisor ;*
- (3.) *To write after these the letters which enter into the dividend and not into the divisor, with their respective exponents.*

Hence

$$\frac{48 a^3 b^3 c^2 d}{12 a b^2 c} = 4 a^2 b c d; \quad \frac{150 a^5 b^3 c d^3}{30 a^3 b^5 d^2} = 5 a^2 b^{-2} c d.$$

23. The division of one simple quantity by another is impossible,

- (1.) *If the coefficient of the one be not divisible by the coefficient of the other ;*
- (2.) *If the exponent of any letter be greater in the divisor than in the dividend ;*
- (3.) *If the divisor contain one or more letters which are not in the dividend.*

If either of these three things happen, the quotient remains under the form of a fraction, that is, an expression in which the algebraical sign of division necessarily enters, but which may often be simplified.

For example, let it be proposed to divide $12 a^4 b^2 c d$ by $8 a^2 b c^2$.

We cannot give this quotient the form of a simple quantity, that is, of a quantity freed from the sign of division, because 12 is not divisible by 8, and the exponent of c is less in the dividend than in the divisor. So the quotient must have the form $\frac{12 a^4 b^2 c d}{8 a^2 b c^2}$. But

this may be simplified ; for, since 4, a^2 , b , and c are common to both terms of the fraction, they may be cancelled, and the result becomes $\frac{3 a^2 b d}{2 c}$.

To simplify a fraction, the terms of which are simple quantities, we must,

- (1.) Divide the two coefficients by their greatest common factor ;
- (2.) Take the least of the two exponents of the same letter from the greatest, and write the letter, with this difference for its exponent, in that term of the fraction which has the greatest exponent ;
- (3.) Write those letters with their respective exponents which are not common to both terms, in the terms to which they belong.

Thus, we find that

$$\frac{48 a^3 b^2 c d^3}{36 a^2 b^3 c^2 d e} = \frac{4 a d^2}{3 b c e} ; \quad \frac{37 a b^3 c^5 d}{6 a^3 b c^4 d^2} = \frac{37 b^2 c}{6 a^2 d} ; \quad \frac{7 a^2 b}{14 a^3 b^3} = \frac{1}{2 a b^2}.$$

In the last example, as all the factors of the dividend are found in the divisor, both terms of the fraction may be divided by the numerator, and the numerator is reduced to unity.

24. It often happens that the exponent of certain letters is the same in the dividend and divisor.

For example, in dividing $24 a^3 b^2$ by $8 a^2 b^2$, as the letter b has the same exponent in both, the quotient must be without b ; and $\frac{24 a^3 b^2}{8 a^2 b^2} = 3 a$. But this quotient may be put in such a form as to preserve traces of the letter b , which has disappeared in the reduction.

If we agree to apply the rule of exponents (22) to the expression $\frac{b^2}{b^2}$ it becomes $\frac{b^2}{b^2} = b^{2-2} = b^0$. This new symbol b^0 indicates (2) that the letter enters 0 times as a factor into the quotient, or, which is the same thing, that it does not enter at all ; but it indicates at the same time that it did enter both into the dividend and divisor, and that it has disappeared in the division.

This symbol has the advantage of preserving traces of a number which made part of the problem to be solved, without changing, on that account, the value of the result ; for, since b^0 came from $\frac{b^2}{b^2}$ which is equal to unity, it follows that $3 a b^0$ is equivalent to $3 a \times 1$, or to $3 a$. In like manner

$$\frac{15 a^2 b^3 c^2}{3 a^2 b c^2} = 5 a^0 b^2 c^0 = 5 b^2.$$

As it is important to have exact notions concerning the origin and signification of the symbols used in algebra, we shall show that generally any quantity a , with an exponent 0, is equivalent to unity, that is, $a^0 = 1$.

This expression arises from a having the same exponent in the dividend and divisor of the division. So we have $a^0 = \frac{a^m}{a^m}$ (where m denotes the whole number which is the exponent of a). But the quotient of a quantity divided by itself is 1. Therefore

$$\frac{a^m}{a^m} = a^0 = 1.$$

We repeat that the symbol a^0 is employed *conventionally* in order to preserve a letter which entered into the enunciation of a question, but which must disappear in the division; and it is often necessary to preserve this mode of expression.

Division of Two Polynomials.

25. It is required to divide

$$51a^3b^3 + 10a^4 - 48a^3b - 15b^4 + 4ab^3$$

by

$$4ab - 5a^2 + 3b^2.$$

In order to follow the steps of the process easily, they are disposed thus;

$$\begin{array}{r}
 51a^3b^3 + 10a^4 - 48a^3b - 15b^4 + 4ab^3 \quad \left. \vphantom{\begin{array}{l} 51a^3b^3 + 10a^4 - 48a^3b - 15b^4 + 4ab^3 \\ 8a^3b - 10a^4 + 6a^2b^2 \end{array}} \right\} \begin{array}{l} 4ab - 5a^2 + 3b^2 \\ -2a^3 + 8ab - 5b^2 \end{array} \\
 \hline
 8a^3b - 10a^4 + 6a^2b^2 \\
 \hline
 57a^3b^3 - 40a^3b - 15b^4 + 4ab^3 \\
 - 32a^3b^3 + 40a^3b - 24ab^3 \\
 \hline
 25a^3b^3 - 15b^4 - 20ab^3 \\
 20ab^3 - 25a^2b^3 + 15b^4 \\
 \hline
 0
 \end{array}$$

The object of this operation is, as we have already said (22), to find a third polynomial, which, being multiplied by the second, shall produce the first.

From this definition and the rule for the multiplication of polynomials, it appears that the dividend is the sum *after addition and reduction*, of the partial products arising from multiplying each term of the divisor by each term of the quotient sought. If then we can discover in the dividend, a term which arose, *without reduc-*

tion, from the multiplication of one of the terms of the divisor by one of the terms of the quotient, then, by dividing the term of the dividend by that of the divisor, we may be certain of obtaining one term of the quotient sought.

Now, from the third remark of article 18., the term $10a^4$ containing the highest exponent of the letter a arises without reduction from the multiplication of those two terms of the divisor and quotient which have the highest exponent of the same letter. Then the division of $10a^4$ by $-5a^2$, certainly gives one term of the quotient sought. But here a difficulty presents itself, which is, to determine the sign which this term of the quotient ought to have. That we may not be hereafter detained on this subject, we proceed to establish a rule for the signs in division.

As in multiplication the product of two terms of the same sign has the sign $+$, and the product of two terms of contrary signs has the sign $-$, we may conclude,

- (1.) That if the term of the dividend has the sign $+$, and the term of the divisor has the sign $+$ also, the term of the quotient must have the sign $+$;
- (2.) If the term of the dividend has the sign $+$, and that of the divisor the sign $-$, the term of the quotient must have the sign $-$, because there is no sign but $-$, which, when combined with the sign $-$ of the divisor, will give the sign $+$ of the dividend;
- (3.) If the term of the dividend has the sign $-$, and the term of the divisor the sign $+$, the quotient must have the sign $-$;
- (4.) If the dividend has the sign $-$, and the divisor the sign $-$, the quotient must have the sign $+$.

That is, if the terms of the dividend and divisor have the same sign, the quotient must have the sign $+$, and if they have contrary signs, the quotient must have the sign $-$. We may abbreviate the rule by saying,

*like signs give $+$,
unlike signs $-$.*

In the proposed example, $10a^4$ and $-5a^2$ have unlike signs, and their quotient must have the sign $-$; moreover $10a^4$, divided by $5a^2$, gives $2a^2$ (22); then $-2a^2$ is a term of the quotient sought. Write this term underneath the divisor, and multiply each term of the divisor by this term, then subtract the product

$$\begin{array}{r} -8a^2b + 10a^4 - 6a^2b^2 \end{array}$$

from the dividend, which is done by writing it with contrary signs underneath the dividend, and performing the reduction. The first partial operation gives

$$57 a^2 b^2 - 40 a^3 b - 15 b^4 + 4 a b^3.$$

This result is made up of the partial products of each of the terms of the divisor, by each of the remaining terms of the quotient. We may then regard it as a new dividend, and reason on it as on the former one. We must then take, in this result, the term $-40 a^3 b$, which has the highest exponent of a , and divide it by the same term $-5 a^2$ of the divisor. Now, by the preceding principles, $-40 a^3 b$ divided by $-5 a^2$ gives $+8 a b$, a new term of the quotient, which is written by the side of the first. Multiply each of the terms of the divisor by this term, and write the product with contrary signs, underneath the second dividend. Make the reduction, and the result of this second operation is

$$25 a^2 b^2 - 15 b^4 - 20 a b^3.$$

Divide $25 a^2 b^2$ by $-5 a^2$, the result is $-5 b^2$, which is the third term of the quotient. Multiply the divisor by this term, write the terms of the product with contrary signs underneath the last dividend, and make the reduction; the result obtained is 0. Then $-2 a^2 + 8 a b - 5 b^2$, or $8 a b - 2 a^2 - 5 b^2$, is the quotient required, which may be verified by multiplying the divisor by this polynomial; the product will be equal to the dividend.

If we reflect on the preceding reasoning, we shall perceive that, as in each partial operation, it is necessary to find that term of the dividend which has the highest exponent of one of the letters, and to divide it by that term of the divisor which has the highest exponent of the same letter, the trouble of looking for them might be avoided by taking care at first to write down the terms of the dividend and divisor, so that the exponents of the same letter should diminish as we go from right to left. This is called *arranging* the dividend and the divisor according to the powers of the same letter. By this preparation, the first terms of the dividend and divisor, on the left hand, are always those which must be divided by one another, in order to obtain one of the terms of the quotient; and it is the same in all the following operations, because the partial quotients, and the products of the divisor by those quotients, are always arranged.

The following is an exhibition of the preceding division, after the arrangement of the two polynomials.

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$$\begin{array}{r}
 10a^4 - 48a^3b + 51a^2b^2 + 4ab^3 - 15b^4 \quad \left. \vphantom{10a^4 - 48a^3b + 51a^2b^2 + 4ab^3 - 15b^4} \right\} \begin{array}{l} - 5a^2 + 4ab + 3b^2 \\ - 2a^2 + 8ab - 5b^2 \end{array} \\
 - 10a^4 + 8a^3b + 6a^2b^2 \\
 \hline
 - 40a^3b + 57a^2b^2 + 4ab^3 - 15b^4 \\
 + 40a^3b - 32a^2b^2 - 24ab^2 \\
 \hline
 25a^2b^2 - 20ab^3 - 15b^4 \\
 - 25a^2b^2 + 20ab^3 + 15b^4 \\
 \hline
 0
 \end{array}$$

26. From this we may deduce the following rule for the division of polynomials; *arrange the dividend and divisor according to the powers of the same letter, divide the first term on the left of the dividend by the first term on the left of the divisor, and the first term of the quotient is found; multiply the divisor by this term, and subtract the product from the dividend. Divide the first term of this remainder by the first term of the divisor, and the second term of the quotient is found; multiply the divisor by this second term, and subtract the product from the result of the first operation. Continue the same series of operations until the result becomes 0; in which case the division is said to be exact.*

When the first term of the arranged dividend is not exactly divisible by the first term of the arranged divisor, it is a sign that the division is impossible, that is, that there is no polynomial, which when multiplied by the divisor can produce the dividend. And, in general, the division is impossible when the first term of one of the partial dividends is not divisible by the first term of the divisor.

27. Though there is some analogy between arithmetical and algebraical division, with respect to the manner in which the operations are disposed and performed, yet there is this essential difference between them, that in arithmetical division the figures of the quotient are obtained by trial, while in algebraic division the quotient obtained by dividing the first term of the dividend by the first term of the divisor, is always one of the terms of the quotient sought. If these two terms are not divisible by one another, we may conclude immediately that the division is impossible. Besides, there is no reason why the operation should not commence from the right instead of the left, and it would then begin with those terms which have the least exponent of the letter, with reference to which the whole has been arranged. In arithmetical division, the quotient can only be found by beginning from the left.

Such is the independence of the partial operations belonging to this process, that after having subtracted from the whole dividend, the product of the divisor by the first term of the quotient, we may, in the second operation, divide by one another those two terms of the new dividend and divisor, which have the highest exponent of a different letter from that which was first taken; and one of the terms of the quotient which remained to be determined is thus found. If we retain the same letter, it is because there is no reason for changing, and because the two polynomials being already arranged with reference to the powers of one letter, the first terms on the left of the dividend and divisor will give a term of the quotient; whereas if the letter were changed, we should have to search again for the terms which contain the highest exponent of the new letter.

28. To divide

$$\begin{array}{l} 21x^3y^2 + 25x^2y^3 + 68xy^4 - 40y^5 - 56x^5 - 18x^4y \\ \text{by} \qquad \qquad \qquad 5y^2 - 8x^2 - 6xy. \end{array}$$

The operations are as follows, after arranging the whole with reference to the powers of y .

$$\begin{array}{r} -40y^5 + 68xy^4 + 25x^2y^3 + 21x^3y^2 - 18x^4y - 56x^5 \\ +40y^5 - 48xy^4 - 64x^2y^3 \qquad \qquad \qquad \left. \begin{array}{l} 5y^2 - 6xy - 8x^2 \\ -8y^3 + 4xy^2 - 3x^2y + 7x^3 \end{array} \right\} \\ \hline 20xy^4 - 39x^2y^3 + 21x^3y^2 - 18x^4y - 56x^5 \\ -20xy^4 + 24x^2y^3 + 32x^3y^2 \\ \hline -15x^2y^3 + 53x^3y^2 - 18x^4y - 56x^5 \\ +15x^2y^3 - 18x^3y^2 - 24x^4y \\ \hline 35x^3y^2 - 42x^4y - 56x^5 \\ -35x^3y^2 + 42x^4y + 56x^5 \\ \hline 0 \end{array}$$

As it is important for beginners to make themselves familiar, with algebraical operations, and, above all, to calculate with readiness, we shall give this last example again with some simplifications, which it is convenient to introduce.

$$\begin{array}{r} -40y^5 + 68xy^4 + 25x^2y^3 + 21x^3y^2 - 18x^4y - 56x^5 \left. \begin{array}{l} 5y^2 - 6xy - 8x^2 \\ -8y^3 + 4xy^2 - 3x^2y + 7x^3 \end{array} \right\} \\ \text{1st rem. } 20xy^4 - 39x^2y^3 + 21x^3y^2 - 18x^4y - 56x^5 \\ \hline \text{2d rem.} \quad -15x^2y^3 + 53x^3y^2 - 18x^4y - 56x^5 \\ \hline \text{3d rem.} \quad \quad 35x^3y^2 - 42x^4y - 56x^5 \\ \hline 0 \end{array}$$

For example, in the expression

$$-15b^2a^2 + 7bc a^2 - 8c^2 a^2,$$

the modification consists in putting this expression under the form

$$-(15b^2 - 7bc + 8c^2)a^2,$$

which, by the second method, is written thus,
$$\begin{array}{r|l} -15b^2 & a^2; \\ +7bc & \\ -8c^2 & \end{array}$$

and this method has the advantage of preserving to each term the sign which it had at first.

In like manner $-19abc + 3ab^2$ is written
$$\begin{array}{r|l} +3b^2 & a. \\ -19bc & \end{array}$$

The operation is performed thus;

$$\begin{array}{r|l|l|l} 10a^3 + 11b & a^3 + 3b^3 & a - 5b^2c + 15bc^2 & \left. \begin{array}{l} 5a^3 + 3ba - 5bc \\ 2a + b - 3c \end{array} \right\} \\ -15c & -19bc & & \\ \hline \text{1st rem.} & 5b & a^2 + 3b^2 & a - 5b^2c + 15bc^2 \\ & -15c & -9bc & \\ \hline \text{2d rem.} & & 0 & \end{array}$$

First, divide $10a^3$ by $5a^2$, and the quotient is $2a$. Subtract the product of the divisor by $2a$, and the first remainder is obtained. Divide the part of this remainder, which contains a^2 , by $5a^2$, and the quotient is $b - 3c$. The subtraction of the divisor, multiplied by $b - 3c$, gives no remainder, therefore $2a + b - 3c$ is the quotient required.

To explain generally the preceding method; which is the most complicated in division, let the dividend be

$$Aa^4 + Ba^3 + Ca^2 + Da + E, \text{ and the divisor } A'a^2 + B'a + c'.$$

(It is customary in algebra, when there are a great number of quantities in the same question, to designate a certain number of them by different letters; and, in order to avoid having too many letters, the others are designated by the same letters accented.)

In these two polynomials, each of the coefficients, $A, B, C, D, E, A', B', c'$, is the aggregate of several terms. Thus, Aa^4 represents all the part of the dividend which contains a^4 . Since the highest exponent of a is 4 in the dividend, and 2 in the divisor, it must be 2 in the quotient, which is therefore of the form

$$A''a^2 + B''a + c''.$$

To determine that part of the quotient which has the highest power of a , observe that the product of the two parts $A'a^2$ and $A''a^2$,

cannot undergo any reduction with the other terms of the product of the divisor and quotient (Remark 3d, art. 18), and therefore must be equal to the part Δa^4 of the dividend which has the highest power of a . If then we divide Δa^4 by $\Delta' a^2$, we shall have the part of the quotient $\Delta'' a^2$, which amounts to dividing Δ by Δ'' , since a^4 , divided by a^2 , gives a^2 . If Δ and Δ' are themselves polynomials of one or more letters, the operation of division is performed, which requires that the polynomials should be arranged according to the powers of some one of their letters; and this is the reason for what is said above, that in writing the terms which are multiplied by one power of some letter, it is necessary to arrange them according to the powers of some other letter. If several terms in one column contain the same exponent of the second letter, they must be arranged according to the powers of a third letter.

Having obtained $\Delta'' a^2$, multiply the divisor by it, and subtract the product. The first remainder is thus obtained, and the same operation is performed on it as on the dividend.

The following are examples of this method. The partial divisions which the principal operations require, are annexed.

$\begin{array}{r} 12b^2 \\ - 29bc \\ + 15c^2 \end{array}$	$\begin{array}{r} a^3 + 23b^3 \\ - 31b^2c \\ - 9bc^2 \\ + 15c^3 \end{array}$	$\begin{array}{r} a^2 + 10b^4 \\ - 6b^2c^2 \end{array}$	$\begin{array}{r} a \end{array}$	$\left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{array}{l} 3b \\ - 5c \\ 4b \\ - 3c \end{array}$	$\left. \begin{array}{l} a + 2b^3 \\ a^2 + 5b^2 \\ - 3c^2 \end{array} \right a$
1st rem.	$\begin{array}{r} + 15b^3 \\ - 25b^2c \\ - 9bc^2 \\ + 15c^3 \end{array}$	$\begin{array}{r} a^2 + 10b^4 \\ - 6b^2c^2 \end{array}$	a		
2d rem.	0				

First partial division.

$\begin{array}{r} 12b^2 - 29bc + 15c^2 \\ - 9bc + 15c^2 \end{array}$	$\left. \begin{array}{l} \\ \end{array} \right\} \begin{array}{l} 3b - 5c \\ 4b - 3c \end{array}$
1st rem.	
2d rem.	0

Second partial division.

$\begin{array}{r} 15b^3 - 25b^2c - 9bc^2 + 15c^3 \\ - 9bc^2 + 15c^3 \end{array}$	$\left. \begin{array}{l} \\ \end{array} \right\} \begin{array}{l} 3b - 5c \\ 5b^2 - 3c^2 \end{array}$
1st rem.	
	0

$$\begin{array}{r|l}
 \begin{array}{r}
 6b|a^4-7b^3|a^3-3b^3|a^2+4b^3|a+b^3 \\
 -10|+23b|+22b^2|-9b^3|-2b \\
 -20|-31b|+5b|-5 \\
 +5|-5
 \end{array}
 & \left. \begin{array}{l}
 3b|a+b^3| \\
 -5|-2b| \\
 2a^3-3b|a^2+4b|a+1 \\
 +4|-1|
 \end{array} \right\}
 \end{array}$$

$$\begin{array}{l}
 \text{1st rem.} \quad \begin{array}{r} -9b^3|a^3* \\ +27b \\ -20 \end{array} \\
 \text{2d rem.} \quad \begin{array}{r} 12b^3|a^3 \\ -23b \\ +5 \end{array} \\
 \text{3d rem.} \quad \begin{array}{r} 3b|a \\ -5 \end{array} \\
 \text{4th rem.} \quad 0
 \end{array}$$

Partial divisions.

$$\begin{array}{l}
 (1.) \quad \begin{array}{r} 6b-10 \\ 0 \end{array} \left. \vphantom{\begin{array}{r} 6b-10 \\ 0 \end{array}} \right\} \begin{array}{r} 3b-5 \\ 2 \end{array} \\
 (2.) \quad \begin{array}{r} -9b^3+27b-20 \\ 12b-20 \\ 0 \end{array} \left. \vphantom{\begin{array}{r} -9b^3+27b-20 \\ 12b-20 \\ 0 \end{array}} \right\} \begin{array}{r} 3b-5 \\ -3b+4 \end{array} \\
 (3.) \quad \begin{array}{r} 12b^2-23b+5 \\ -3b+5 \\ 0 \end{array} \left. \vphantom{\begin{array}{r} 12b^2-23b+5 \\ -3b+5 \\ 0 \end{array}} \right\} \begin{array}{r} 3b-5 \\ 4b-1 \end{array} \\
 (4.) \quad \begin{array}{r} 3b-5 \\ 0 \end{array} \left. \vphantom{\begin{array}{r} 3b-5 \\ 0 \end{array}} \right\} \begin{array}{r} 3b-5 \\ 1 \end{array}
 \end{array}$$

30. There is another important case of algebraic division; where the dividend contains one or more letters which the divisor does not contain. The division may be made in the ordinary manner, but there is a much more simple method of finding the quotient. Suppose, for example, that the dividend contains various powers of the letter a , and that this letter does not enter into the divisor (which is then said to be *independent* of a). Arrange the dividend according to the powers of a , and let it be

$$Aa^4 + Ba^3 + Ca^2 + Da + E,$$

4 being the highest exponent of a , and A, B, C, D, E , simple quantities or polynomials not containing a . Let m be the divisor, which is independent of a .

Since the divisor, multiplied by the quotient, must reproduce the

* The remainders are not written down at full length, as it would only be repeating the terms of the dividend.

dividend, and since the divisor m does not contain a , it is clear that the quotient is a polynomial, which contains the same powers of a as are in the dividend. The quotient is therefore necessarily of the form

$$A' a^4 + B' a^3 + C' a^2 + D' a + E'.$$

Imagine that this quotient is found, and that the divisor is multiplied by the several parts $A' a^4$, $B' a^3$, &c. The products will be $A' m a^4$, $B' m a^3$, $C' m a^2$, &c.; and as these are not capable of any reduction with one another, since they contain different powers of a , they must be respectively equal to the terms $A a^4$, $B a^3$, &c., of the dividend.

We have therefore,

$$A' m = A, \text{ or } A' = \frac{A}{m},$$

$$B' m = B, \text{ or } B' = \frac{B}{m},$$

$$\&c. \qquad \&c.$$

and so on; whence this general proposition is deduced;

If a polynomial, arranged according to the powers of a certain letter, be exactly divisible by a polynomial which is independent of that letter, each of the coefficients of the various powers which enter into the first polynomial, must be divisible by the second. The coefficients of the various powers of that letter in the quotient are quotients arising from the division of the coefficients of the dividend by the divisor.

To divide

$$3a^2b^3 - 3abc^3 - 2b^3c^2 + b^5 - 3a^2bc^2 + 3ab^3c - a^2c^3 + bc^4 + a^2b^2c$$

by $b^3 - c^3$.

The dividend, arranged with reference to a , is

$$(3b^3 + b^2c - 3bc^2 - c^3)a^2 + (3b^3c - 3bc^3)a + b^5 - 2b^3c^2 + bc^4.$$

The three partial divisions are

$$\frac{3b^3 + b^2c - 3bc^2 - c^3}{b^3 - c^3}, \quad \frac{3b^3c - 3bc^3}{b^3 - c^3}, \quad \text{and} \quad \frac{b^5 - 2b^3c^2 + bc^4}{b^3 - c^3},$$

the quotients of which are $3b + c$, $3bc$, and $b^3 - bc^2$; therefore the whole quotient is

$$(3b + c)a^2 + 3bca + b^3 - bc^2.$$

The two last quotients can be obtained more easily than by the ordinary method, if we observe that

$$3b^3c^2 - 3bc^3 = 3bc(b^2 - c^2),$$

and

$$b^5 - 2b^3c^2 + bc^4 = b(b^4 - 2b^2c^2 + c^4) = b(b^2 - c^2)^2 \quad (19).$$

We may observe, while on this subject, that though there are general rules for performing all these operations, they can very often be simplified, and these simplifications must not be neglected when there is an opportunity of making them.

31. Among the different examples of algebraic division, there is one of such remarkable application, and so often met with in the resolution of problems, that algebraists have made a kind of theorem of it. We have seen (5 and 19), that

$$(a + b)(a - b) = a^2 - b^2;$$

hence, $\frac{a^2 - b^2}{a - b}$ gives $a + b$ for its quotient.

Divide $a^3 - b^3$ by $a - b$, and the quotient is $a^2 + ab + b^2$. In like manner, divide $a^4 - b^4$ by $a - b$, and the quotient is

$$a^3 + a^2b + ab^2 + b^3.$$

These results are obtained by the ordinary operations of division; and analogy would lead us to conclude that, however great the exponents of a and b , the division leaves no remainder; but analogy cannot lead to certainty or rigour.

To be certain on this point, call m the exponent of a and b , and begin the division of $a^m - b^m$ by $(a - b)$. First divide a^m by a , the quotient is a^{m-1} (22). The product of $a - b$ and a^{m-1} , being subtracted from the dividend, leaves $a^{m-1}b - b^m$ for the first remainder, which may be put under the form $b(a^{m-1} - b^{m-1})$. Whence it is plain, that if $a^{m-1} - b^{m-1}$ be exactly divisible by $a - b$, so also is $a^m - b^m$; that is, if the difference of the same powers of a certain degree of two quantities is divisible by their difference, the difference of the powers of the next greater degree is also divisible by that difference. Now, $\frac{a^2 - b^2}{a - b}$ is exactly divisible, and gives the quotient $a + b$; whence $\frac{a^3 - b^3}{a - b}$ gives an exact quotient, equal to

$$a^2 + \frac{a^2 - b^2}{a - b} \cdot b,$$

or to $a^2 + (a + b)b$, or to $a^2 + ab + b^2$.

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For the same reason $\frac{a^4 - b^4}{a - b}$ gives an exact quotient equal to

$$a^3 + b \frac{a^3 - b^3}{a - b},$$

or to $a^3 + b(a^2 + ab + b^2)$, or to $a^3 + a^2b + ab^2 + b^3$.

and, generally, $\frac{a^m - b^m}{a - b}$ gives an exact quotient equal to

$$a^{m-1} + a^{m-2}b + a^{m-3}b^2 + \dots + ab^{m-2} + b^{m-1}.$$

This proposition may be verified *à posteriori*, by performing the multiplication

$$(a^{m-1} + a^{m-2}b + a^{m-3}b^2 + \dots + ab^{m-2} + b^{m-1})(a - b).$$

We may see that the partial products a^m and $-b^m$ are the only ones which are not destroyed in the reduction. For example, the multiplication of $a^{m-2}b$ by a , gives $a^{m-1}b$, and the multiplication of a^{m-1} by $-b$ gives $-a^{m-1}b$, which destroys the preceding product. The same holds of the other terms.

Beginners should reflect well on the preceding method of demonstration, as it is often employed in algebra.

32. We have given (23 and 26) the principal characteristics by means of which it appears that the division of simple quantities or polynomials cannot be exactly performed; that is, that no third algebraic quantity can be found, which, multiplied by the second, will produce the first.

We may add that, in polynomials, simple inspection often shows that one is not divisible by another. When the polynomials contain two or more letters, before arranging them according to the powers of one of the letters, compare the two terms of the dividend and divisor which contain the highest exponents of each letter. If the terms containing the highest exponents of any one of these letters are not divisible by one another, an exact division is impossible. The same remark applies to each of the operations which the process contains.*

* These remarks are only applicable to all cases in which the coefficients are whole numbers. For it may happen that two polynomials, one or both of which contain fractional coefficients, may be divisible by one another. For instance

$$\frac{3}{10}a^2 + \frac{82}{172}ab + \frac{2}{35}b^2, \text{ divided by } \frac{3}{2}a + \frac{1}{5}b,$$

gives the quotient $\frac{1}{5}a + \frac{2}{7}b.$

Suppose it were required to divide

$$12a^3 - 5a^2b + 7ab^2 - 11b^3$$

by

$$4a^2 - 8ab + 3b^2.$$

With the letter a the division appears possible, but on looking at b it is shown to be impossible, since $-11b^3$ is not divisible by $3b^2$.

The following considerations will conclude this subject ;

(1.) A polynomial cannot be divided by another polynomial containing a letter which is not in the dividend ; for it is impossible that a third quantity, multiplied by a second, which contains a given letter, can produce a result independent of that letter.

(2.) A simple quantity is never divisible by a polynomial, because (18) the product of two polynomials has always two terms at least which cannot be reduced.

(3.) A polynomial is not divisible by a simple quantity, unless that simple quantity divide each term of the dividend exactly, and the quotient is found by separating the common factor from all the terms.

Examples in Division of Compound Quantities.

1. $(4ac - 2ade) \div 2a = 2c - 2de.$
2. $(8a^2 - 6ab) \div -2a = -4a + 3b.$
3. $(ab - ac) \div (b - c) = a.$
4. $(ac - bc + ad - bd) \div (a - b) = c + d.$
5. $(4a^3 + 6ab - 4ax + 9bx - 15x^2) \div (2a + 3x)$
 $= 2a + 3b - 5x.$
6. $(4x^3 + 4x^2 - 29x + 21) \div (2x - 3) = 2x^2 + 5x - 7.$
7. $(36a^2 - 12ab + 4b^2 - 3bac + 12bc + 9c^2) \div (6a$
 $- 2b - 3c)$
 $= 6a - 2b - 3c.$
8. $(a^3 - 4b^4) \div (a^2 + 2b^2) = a - 2b^2.$
9. $(a^4 - 9b^3 - 6abc^2 - c^4) \div (a^2 - 3ab - c^2)$
 $= a^2 + 3ab + c^2.$
10. $(64a^3 + 64ab + 16b^3 - 9d^3 - 48d - 64) \div (8a$
 $+ 4b + 3d + 8)$
 $= 8a + 4b - 3d - 8.$
11. $(32a^5 + b^5) \div (2a + b)$
 $= 16a^4 - 8a^3b + 4a^2b^2 - 2ab^3 + b^4.$

$$\begin{aligned}
 12. \quad & (18a^2 + 33ab + 42ac - 12ad - 30b^2 + 124bc \\
 & + 8bd - 16c^2 - 32cd) \div (6a + 15b - 2c - 4d) \\
 & = 3a - 2b + 8c.
 \end{aligned}$$

Of Algebraic Fractions.

33. Algebraic fractions are the same in their nature as arithmetical fractions; that is, the unit is conceived to be divided into as many equal parts as there are units in the denominator (this denominator being either a simple quantity or a polynomial), and as many of these parts are taken as there are units in the numerator. Hence, addition, subtraction, multiplication, and division of fractions, are performed by the rules which are laid down in arithmetic. But in all applications of these rules, the foregoing methods for treating whole algebraical quantities must be attended to. It would be needless to detain the student on this subject; the sequel will afford abundant opportunities for the practice of these rules.

Nevertheless, *the reduction of algebraic fractions to their most simple expression is entitled to particular notice.*

When the division of simple quantities cannot be exactly performed, it is indicated by the common sign of division; and in this case the quotient takes the form of a fraction, which we have already shown how to simplify (23). Such polynomial fractions as the following are also very easily reduced. As a first example, take

$$\frac{a^2 - b^2}{a^2 - 2ab + b^2}.$$

This fraction (19) can be put under the form

$$\frac{(a-b)(a+b)}{(a-b)^2};$$

suppress the factor $a - b$, which is common to both terms, and the result is

$$\frac{a+b}{a-b}.$$

Again, take

$$\frac{5a^3 - 10a^2b + 5ab^2}{8a^3 - 8a^2b}.$$

This expression is equivalent to

$$\frac{5a(a^2 - 2ab + b^2)}{8a^2(a-b)}, \text{ or to } \frac{5a(a-b)^2}{8a^2(a-b)}.$$

Elementary Theory of the Greatest Algebraic Common Divisor. 37

Suppress the common factor $a(a - b)$, and the result is

$$\frac{5(a - b)}{8a}.$$

These particular cases which we have examined are those in which the terms of the fraction contain the product of the sum and difference of the two quantities, or the square of their sum and difference; practice will enable the student to effect these and similar decompositions with ease, when they are possible.

But the two terms of the fraction may be more complicated polynomials, and then their decomposition into factors is no longer so easy, and recourse must be had to the process of finding the *greatest common divisor*.

Elementary Theory of the Greatest Algebraic Common Divisor.

34. *The greatest common divisor of two polynomials is the greatest polynomial, with reference to the exponents and coefficients which divides exactly the two proposed polynomials.*

The distinguishing property of the greatest common divisor is, that if we divide the two proposed polynomials by it, the quotients which result are *prime to each other*, that is, they do not contain any common factor.

This proposition is evident. For let A and B be the given polynomials, D their greatest common divisor, A' and B' the quotients, we have necessarily

$$A = A' \times D \text{ and } B = B' \times D.$$

Now, if A' and B' had still a common factor d , it would follow that $d \times D$ would be a divisor common to the two polynomials, and greater than D , either with reference to the exponents or to the coefficients, which would be contrary to the definition of D .

35. We have seen (*Arith.* 61),

(1.) *That the greatest common divisor of two whole numbers contains as factors all the particular divisors common to the two numbers, and can contain no others;*

(2.) *That the greatest common divisor of two whole numbers is the same as that of the smaller number and the remainder after their division.*

The theory of the greatest algebraic common divisor depends equally upon these two principles.

This being admitted, suppose in the first place that it is required to find the greatest common divisor of the two polynomials

$$a^3 - a^2 b + 3 a b^2 - 3 b^3 \text{ and } a^2 - 5 a b + 4 b^2.$$

First Operation.

$$\begin{array}{r} a^3 - a^2 b + 3 a b^2 - 3 b^3 \\ + 4 a^2 b - a b^2 - 3 b^3 \end{array} \left. \vphantom{\begin{array}{r} a^3 - a^2 b + 3 a b^2 - 3 b^3 \\ + 4 a^2 b - a b^2 - 3 b^3 \end{array}} \right\} \begin{array}{r} a^3 - 5 a b + 4 b^2 \\ a + 4 b \dots\dots \end{array}$$

1st rem.

$$19 a b^2 - 19 b^3$$

or

$$19 b^2 (a - b)$$

Second Operation.

$$\begin{array}{r} a^2 - 5 a b + 4 b^2 \\ - 4 a b + 4 b^2 \end{array} \left. \vphantom{\begin{array}{r} a^2 - 5 a b + 4 b^2 \\ - 4 a b + 4 b^2 \end{array}} \right\} \begin{array}{r} a - b \\ a - 4 b \end{array}$$

$$\hline 0$$

which gives $a - b$ for the greatest common divisor.

We commence by dividing the polynomial of the higher degree by that of the lower; the quotient is, as we have seen above, $a + 4 b$; and for a remainder we obtain . . . $19 a b^2 - 19 b^3$.

It follows from the second principle, that the greatest common divisor sought is the same as that of the remainder, and the polynomial which we have taken as a divisor.

But since $19 a b^2 - 19 b^3$ may be put under the form . . . $19 b^2 (a - b)$, we see that the factor $19 b^2$ divides this remainder without dividing $a^2 - 5 a b + 4 b^2$; then, according to the first principle, this factor cannot enter into the greatest common divisor; so that we can, without inconvenience, suppress it, and the question is reduced to seeking the greatest common divisor of

$$a^2 - 5 a b + b^2, \text{ and } a - b.$$

Now dividing the first of these two polynomials by the second, we have for an exact quotient, $a - 4 b$; then $a - b$ is the greatest common divisor of the two proposed polynomials.

Let us take again the same example, having arranged it with reference to b ,

$$- 3 b^3 + 3 a b^2 - a^2 b + a^3 \text{ and } 4 b^2 - 5 a b + a^2.$$

First Operation.

$$\begin{array}{r}
 \begin{array}{r}
 -12b^3 + 12ab^2 - 4a^2b + 4a^3 \\
 -3ab^2 - a^2b + 4a^3 \\
 \hline
 -12ab^2 - 4a^2b + 16a^3
 \end{array}
 \left. \vphantom{\begin{array}{r} -12b^3 + 12ab^2 - 4a^2b + 4a^3 \\ -3ab^2 - a^2b + 4a^3 \end{array}} \right\} \begin{array}{l} 4b^3 - 5ab + a^2 \\ -3b, -3a \end{array} \\
 \text{1st rem.} \\
 \hline
 \begin{array}{r}
 -19a^2b + 19a^3 \\
 19a^2(-b + a)
 \end{array} \\
 \text{2d rem.} \\
 \text{or}
 \end{array}$$

Second Operation.

$$\begin{array}{r}
 4b^3 - 5ab + a^2 \left. \vphantom{4b^3 - 5ab + a^2} \right\} \begin{array}{l} -b + a \\ -4b + a \end{array} \\
 -ab + a^2 \\
 \hline
 0
 \end{array}$$

which gives $-b + a$ or $a - b$ for the greatest common divisor.

At first it is difficult to perform the division of the two polynomials because the first term $-3b^3$ of the dividend is not divisible by the first term $4b^2$ of the divisor. But if we observe that the coefficient 4 is not a factor of all the terms of $4b^3 - 5ab + a^2$, and that thus, according to the first principle, 4 cannot be a part of the greatest common divisor, we can without any inconvenience introduce this factor into the dividend which gives

$$-12b^3 + 12ab^2 - 4a^2b + 4a^3;$$

then the division of the two first terms becomes possible.

Performing this division we find for a quotient $-3b$, and for a remainder $-3ab^2 - a^2b + 4a^3$.

As in this remainder, the exponent of b is still equal to that of the divisor, we may continue the division by multiplying anew this remainder by 4, in order to render the division of the two first terms possible.

This being done, the expression becomes

$$-12ab^2 - 4a^2b + 16a^3,$$

which, divided by $4b^2 - 5ab + a^2$, gives for a quotient $-3a$ (which we separate from the first by a comma, as having no connexion with it), and for a remainder $-19a^2b + 19a^3$.

This last remainder can be put under the form $19a^2(-b + a)$. We then suppress the factor $19a^2$, as making no part of the greatest common divisor, and the question is reduced to finding the greatest common divisor of $4b^2 - 5ab + a^2$ and $-b + a$.

Dividing these two polynomials by each other, we find for an exact quotient $-4b + a$; then $-b + a$ or $a - b$ is the greatest common divisor sought.

36. In this example, as in all others where the exponent of the principal letter is greater by unity in the dividend than in the divisor, we can abridge the operation by multiplying each term of the dividend by the square of the coefficient of the first term of the divisor. We see indeed, by this means, that the first partial quotient which we obtain, ought to contain the first power of this coefficient. By multiplying the divisor by the quotient, and making the reduction, with the dividend thus prepared, we have a result which ought still to contain the coefficient as a factor, and the division can be continued until we obtain a remainder of a lower degree than the divisor, with reference to the principal letter.

The following are the operations ;

First Operation.

Having multiplied by 16 or by the square of 4 ;

$$\begin{array}{r}
 -48b^3 + 48ab^2 - 16a^2b + 16a^3 \} 4b^3 - 5ab + a^2 \\
 -12ab^2 - 4a^2b + 16a^3 \} \underline{12b - 3a} \\
 \text{1st rem.} \quad -19a^2b + 19a^3 \\
 \text{or} \quad 19a^2(-b + a)
 \end{array}$$

Second Operation.

$$\begin{array}{r}
 4b^3 - 5ab + a^2 \} -b + a \\
 -ab + a^2 \} \underline{-4b + a} \\
 0
 \end{array}$$

N. B. If the exponent of the principal letter in the dividend exceeds the exponent of the same letter in the divisor by two or three units, we must multiply the dividend by the third or fourth power of the coefficient of the first term of the divisor. This is easily perceived.

37. Let there be for a second example,

$$\begin{array}{l}
 15a^5 + 10a^4b + 4a^3b^2 + 6a^2b^3 - 3ab^4 \\
 \text{and} \quad 12a^3b^2 + 38a^2b^3 + 16ab^4 - 10b^5.
 \end{array}$$

Before proceeding with the division of these two polynomials, we begin by observing that the first has a common factor a in all its terms ; and since this factor is not found in the second polynomial, we may suppress it, as making no part of the common divisor.

For the same reason the factor $2b^2$, common to all the terms of the second polynomial, and not found in the first, may be suppressed. So that the question is reduced to seeking the greatest common divisor of the polynomials

$$15a^4 + 10a^3b + 4a^2b^2 + 6ab^3 - 3b^4$$

and $6a^3 + 19a^2b + 8ab^2 - 5b^3$.

First Operation.

$$\begin{array}{r} 30a^4 + 20a^3b + 8a^2b^2 + 12ab^3 - 6b^4 \\ - 75a^3b - 32a^2b^2 + 37ab^3 - 6b^4 \\ \hline - 150a^3b - 64a^2b^2 + 74ab^3 - 12b^4 \end{array} \quad \left. \begin{array}{l} 6a^3 + 19a^2b + 8ab^2 - 5b^3 \\ 5a, \quad -25b \end{array} \right\}$$

1st rem. $+ 411a^2b^2 + 274ab^3 - 137b^4$;

or $137b^2(3a^2 + 2ab - b^2)$.

Second Operation.

$$\begin{array}{r} 6a^3 + 19a^2b + 8ab^2 - 5b^3 \\ + 15a^2b + 10ab^2 - 5b^3 \\ \hline 0 \end{array} \quad \left. \begin{array}{l} 3a^2 + 2ab - b^2 \\ 2a + 5b \end{array} \right\}$$

Then $3a^2 + 2ab - b^2$ is the greatest common divisor.

By following the same method as in the preceding example, we must multiply the whole dividend by the coefficient 6 of the first term of the divisor, or rather by the square of six; but as 15 and 6 have a common factor 3, it is evidently sufficient to multiply the whole dividend by 2, the factor of 6, which does not enter into 15.

This being done, we perform the division which gives in the first place a remainder, whose first term is $-75a^3b$. As 75 still contains the factor 3, which enters into 6, it is sufficient to multiply this remainder by 2, in order to continue the division which, being performed, we have the first *principal remainder*

$$411a^2b^2 + 274ab^3 - 137b^4.$$

Now it is easy to perceive that in this remainder there is still a common factor $137b^2$; and since this factor does not enter into the second polynomial, we may suppress it as making no part of the common divisor, and the question is reduced to seeking the greatest common divisor of the two polynomials

$$6a^3 + 19a^2b + 8ab^2 - 5b^3$$

and $3a^2 + 2ab - b^2$.

By performing the division of these two polynomials, we find for an

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exact quotient $2a + 5b$; so that the remainder $3a^2 + 2ab - b^2$ is the greatest common divisor sought.

38. *Remark.* It may be asked if the suppression of factors common to all the terms of one of the remainders which, we have made in the course of the calculation, have no other object than simplifying these calculations, or if these operations are indispensable. Now it will be readily perceived, that these suppressions are necessary; for, if in the preceding example, we had not suppressed the factor, $37b^2$, it would have been necessary, in order to render the division of the first term of the new dividend by the first term of the divisor possible, to multiply the whole dividend by $137b^2$; but then we should introduce into the dividend a factor which is found likewise in the divisor, whence it follows, that the greatest common divisor sought is combined with the factor $137b^2$, which ought to make no part of it.

The following example is suited to confirm what we have just said.

39. Let it be required to find the greatest common divisor of the two polynomials

$$\begin{array}{l} ab + 2a^2 - 3b^2 - 4bc - ac - c^2 \\ \text{and} \quad 9ac + 2a^2 - 5ab + 4c^2 + 8bc - 12b^2. \end{array}$$

First Operation.

$$\begin{array}{r} 2a^2 + b \mid a - 3b^2 \\ -c \mid -4bc \\ \hline c^2 \end{array} \left. \begin{array}{r} 2a^2 - 5b \mid a - 12b^2 \\ + 9c \mid 8bc \\ + 4c^2 \end{array} \right\} \begin{array}{r} 1 \\ \hline \end{array}$$

$$\begin{array}{r} \text{1st. rem.} \quad 6b \mid a + 9b^2 \\ -10c \mid -12bc \\ 5c^2; \end{array}$$

$$\text{or} \quad (3b - 5c)(2a + 3b + c).$$

Second Operation.

$$\begin{array}{r} 2a^2 - 5b \mid a - 12b^2 \\ + 9c \mid + 8bc \\ + 4c^2 \end{array} \left. \begin{array}{r} 2a + 3b + c \\ \hline a - 4b \\ + 4c \end{array} \right\}$$

$$\begin{array}{r} -8b \mid a - 12b^2 \\ + 8c \mid + 8bc \\ + 4c^2 \end{array}$$

0

Therefore $2a + 3b + c$ is the greatest common divisor.

After having arranged the two polynomials, we can, without any preparation, perform the division, which gives for the first remainder,

$$\begin{array}{r|l} 6b & a + 9b^2 \\ -10c & -12bc \\ \hline & -5c^2. \end{array}$$

In order to continue the operation it would be necessary, by taking the second polynomial for a dividend, and this remainder for a divisor, to multiply the new dividend by $6b - 10c$, or simply by $3b - 5c$, because the factor 2 belongs already to the first term of the dividend; but before performing the multiplication, let us see if this factor $3b - 5c$, will not divide the second term of the remainder, namely, $9b^2 - 12bc - 5c^2$. Now this division succeeds, and gives for an exact quotient $3b - c$; whence it follows, that the remainder may be put under the form

$$(3b - 5c)(2a + 3b + c).$$

As the factor $3b - 5c$ is found in this remainder, and does not belong to the new dividend, (since this factor, being independent of the letter a , must (30) be found among the coefficients of the different powers of this letter, which is not the case,) we can, without inconvenience, suppress this factor.

This suppression is moreover indispensable, because, without it, we must introduce this factor into the dividend; and then the two polynomials, containing a common factor, which they had not before, the greatest common divisor would be changed; it would involve the factor $3b - 5c$, which makes no part of it.

The suppression being made, we perform the new division, which gives an exact quotient; therefore $2a + 3b + c$ is the greatest common divisor.

40. We propose, as the last example, to find the greatest common divisor to the two polynomials

$$a^4 + 3a^3b + 4a^2b^2 - 6ab^3 + 2b^4$$

and

$$4a^2b + 2ab^2 - 2b^3,$$

or simply, $2a^2 + ab - b^2$, since the factor $2b$ may be suppressed in the second.

First Operation.

$$\begin{array}{r}
 8a^4 + 24a^3b + 32a^2b^2 - 48ab^3 + 16b^4 \quad \left. \begin{array}{l} 2a^2 + ab - b^2 \\ 4a^2 + 10ab + 13b^2 \end{array} \right\} \\
 + 20a^3b + 36a^2b^2 - 48ab^3 + 16b^4 \\
 + 26a^2b^2 - 38ab^3 + 16b^4
 \end{array}$$

1st. rem.

$$- 51ab^3 + 29b^4,$$

or

$$- b^3(51a - 29b).$$

Second Operation.

Multiplication by 2601 the square of 51.

$$\begin{array}{r}
 5202a^2 + 2601ab - 2601b^2 \quad \left. \begin{array}{l} 51a - 29b \\ 102a + 109b \end{array} \right\} \\
 - 5202a^2 + 2958ab \\
 \hline
 + 5559ab - 2601b^2 \\
 - 5559ab + 3161b^2 \\
 \hline
 + 560b^2.
 \end{array}$$

2d. rem.

$$+ 560b^2.$$

The exponent of the letter a in the dividend exceeding by two units that of the same letter in the divisor, we multiply the whole dividend by the cube of 2, that is, by 8. With this preparation, we perform three successive divisions, and obtain for the first principal remainder $- 51ab^3 + 29b^4$. Suppressing the factor b^3 in this remainder, we have for a new divisor $- 51a + 29b$, or, changing the signs, which we are allowed to do, $51a - 29b$; the new dividend is moreover $2a^2 + ab - b^2$.

Multiplying this dividend by the square of 51, or 2601, then performing the division, we obtain, for the second principal remainder, $+ 560b^2$; which shows that the two proposed polynomials are *prime to each other*, that is, they have not a common factor. Indeed it results from the second principal (35), that the greatest common divisor must be a factor in the remainder of each operation; thus it must divide the remainder $560b^2$; but this remainder is independent of the principal letter a ; therefore, if the two polynomials have a common divisor, it must be *independent of a* , and consequently (30) be found as a factor in the coefficients of the different powers of this letter, which contain each of the two proposed polynomials, and this is evidently not the case.

These examples will suffice to make the learner acquainted with the course to be pursued in finding the greatest common divisor of two polynomials.

41. GENERAL RULE. We begin by suppressing in the two polynomials the simple factors common to the two terms. (It

may happen that the simple factor which is found in the dividend, and that which the divisor contains, have a common divisor; in this case, we put it aside, as making a part of the common divisor sought.) *This suppression being made, we prepare the dividend so as to render possible the division of its first term by that of the divisor (35, 36); then, we perform the division, which gives a certain remainder of a degree less than the divisor, in which we suppress the simple or polynomial factors contained in the coefficients of the different powers of the principal letter. We then take this remainder for a divisor, the second polynomial for a dividend, and perform the same operation with these as with the preceding. This series of operations is to be continued until we obtain a remainder which exactly divides the preceding; then this remainder is the greatest common divisor; even if we obtain a divisor INDEPENDENT of the principal letter, which indicates that the two proposed polynomials are PRIME TO EACH OTHER, except they have a common factor independent of the letter, which factor would not have been discovered at the commencement of the operation.*

The following examples will serve to illustrate the method above given,

$$\begin{array}{l} (1.) \quad \left. \begin{array}{l} q n p^3 + 3 n p^2 q^2 - 2 n p q^3 - 2 n q^4, \\ 2 m p^2 q^2 - 4 m p^4 - m p^3 q + 3 m p q^2. \end{array} \right\} \\ \text{and} \end{array}$$

The greatest common divisor is $p - q$;

$$\begin{array}{l} (2.) \quad \left. \begin{array}{l} 36 a^6 - 18 a^5 - 27 a^4 + 9 a^3, \\ 27 a^5 b^2 - 18 a^4 b^2 - 9 a^3 b^2. \end{array} \right\} \\ \text{and} \end{array}$$

The greatest common divisor is $9 a^3 (a - 1)$.

The theory of the four fundamental operations of algebra and that of the greatest common divisor enables us to perform a great number of questions. We shall hereafter establish new rules as we have occasion for them. We proceed now to the resolution of problems of the first degree.

Examples in the Reduction of Fractional Expressions to their Simplest Terms.

$$1. \quad \frac{ax + x^2}{3bx - cx} = \frac{a + x}{3b - c}.$$

2. $\frac{14a^2 - 7ab}{10ac - 5bc} = \frac{7a}{5c}.$
3. $\frac{12a^3x^4 + 2a^2x^5}{18ab^2x + 3b^2x^2} = \frac{2a^2x^3}{3b^2}.$
4. $\frac{5a^2 + 5ax}{a^2 - x^2} = \frac{5a}{a - x}.$
5. $\frac{a^3 - x^3}{(a - x)^2} = \frac{a^2 + ax + x^2}{a - x}.$
6. $\frac{n^3 - 2n^2}{n^2 - 4n + 4} = \frac{n^2}{n - 2}.$
7. $\frac{x^2 + 2x - 3}{x^2 + 5x - 6} = \frac{x - 1}{x + 2}.$
8. $\frac{2x^3 + 3x^2 + x}{x^3 - x^2 - 2x} = \frac{2x + 1}{x - 2}.$
9. $\frac{a^3b^3 + c^3x^3}{a^2b^2 - c^2x^2} = \frac{a^2b^2 - abcx + c^2x^2}{ab - cx}.$
10. $\frac{2x^3 - (3c + d + 2)x^2 + (3c + d)x}{x^4 - x} = \frac{2x - 3c - d}{x^2 + x + 1}.$
11. $\frac{a^2 + b^2 + c^2 + 2ab + 2ac + 2bc}{a^2 - b^2 - c^2 - 2bc} = \frac{a + b + c}{a - b - c}.$
12. $\frac{a^2 - 3ab + ac + 2b^2 - 2bc}{a^2 - b^2 + 2bc - c^2} = \frac{a - 2b}{a + b - c}.$
13.
$$\begin{aligned} & \frac{(a + b)(a + b + c)(a + b - c)}{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4} \\ &= \frac{(a + b)(a + b + c)(a + b - c)}{4b^2c^2 - (a^2 - b^2 - c^2)^2} \\ &= \frac{a + b}{(c + a - b)(b - a + c)}. \end{aligned}$$

*Problems of the First Degree.**Preliminary Notions on Equations.*

42. THE problems usually treated of in algebra are those whose enunciations, translated into algebraical language, give rise to *equations*. On looking at the solution of the problem of article 3, we

shall see that it is composed of two distinct parts. In the first, the relations which the enunciation of the question establishes between the known and unknown quantities, are written algebraically. In this manner we arrive at the expression of two equal quantities, which is called an equation. Such is the expression $2x + b = a$. In the second part, a series of other equations is deduced from the equation of the problem, of which the last gives the value of the unknown by means of the known quantities. Such is the result

$$x = \frac{a - b}{2},$$

at which we arrived. This is what is called solving the equation.

As the rules to be followed, in putting a problem into an equation, are rather vague, we shall begin with the second part, in which they are fixed and invariable.

From the definition of an equation, it is composed of two parts, separated from one another by the sign $=$. The part on the left is called the *first member*, and that on the right the *second member*.

There are several kinds of equalities.

(1.) The equalities which exist between known numbers represented by letters, such as

$$a - b = c - d, \quad \frac{a}{b} = \frac{c}{d}$$

which may be verified immediately by putting, instead of a, b, c , and d , the particular numbers for which we suppose these equalities to stand.

(2.) Such as are evidently true of themselves, and which can be verified as they actually stand, as

$$25 = 12 + 13; \quad 3a - 5b = a - b + 2a - 4b.$$

These are called *identical* or *verified equalities*.

(3.) Such as cannot be verified except by substituting in place of one or more of the letters, designating unknown quantities, certain numbers whose values depend on the known and given numbers which enter into the equality.

To distinguish this from other equalities, we give it the name of *equation*. It is of this that we now propose to treat.

Equations are divided into different classes; those in which there is only the first power of the unknown quantity are said to be of the *first degree*, such are

$$3x + 5 = 17 - 5x, \quad ax + b = cx + d.$$

The equation

$$2x^2 - 3x = 5 - 2x^2$$

is of the second degree.

The equation

$$4x^3 - 5x^2 + x = 2x^2 + 11$$

is of the third degree.

Generally the degree of an equation is the same as the greatest exponent of the unknown quantity in the equation.

A distinction is made between *numerical* and *literal* equations. The first are those which contain only particular numbers, with the exception of the unknown quantity, which is always designated by a letter. Thus,

$$4x - 3 = 2x + 5, \quad 3x^2 - x = 8,$$

are numerical equations. They are the algebraical translation of problems in which the given quantities are particular numbers.

The equations

$$ax + b = cx + d, \quad ax^2 + bx = c,$$

are literal equations. The given quantities of the problem are here represented by letters. It is customary, in order to distinguish the known and unknown quantities, to denote the latter by the last letters of the alphabet, as x , y , and z .

We proceed to show how to solve an equation of the first degree with one unknown quantity, that is, to find a number, which, being substituted for the unknown quantity in the equation, shall satisfy it; that is, make the first member identically equal to the second.

Of Equations of the First Degree with One Unknown Quantity.

43. We must regard it as an axiom, that we may, without altering the truth of any equation, add the same number to both its members, or subtract it from them; and also, that both members may be multiplied or divided by the same number; that is to say, that if the two members were equal before, they are still equal when changed in the manner expressed above.

The following transformations are of continual use in the solution of equations.

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(1.) When the two members of an equation are simple polynomials, it is generally necessary to transpose some of the terms from one member to the other.

Take the equation $5x - 6 = 8 + 2x$. To disengage x from this equation, we must endeavour to get it by itself in the first member. Subtract $2x$ from both members, and the equation is still true; thus we have $5x - 6 - 2x = 8$. Here we see that the term $2x$, which had the positive sign in the second member, has a negative one in the first.

Secondly, add 6 to both members, and the equation becomes

$$5x - 6 - 2x + 6 = 8 + 6,$$

or, since the terms -6 , $+6$, destroy each other,

$$5x - 2x = 8 + 6.$$

Then the term which was negative in the first member, takes a positive sign when it passes to the second member.

Take the equation $ax + b = d - cx$. Add cx to and take b from both members, and it becomes

$$ax + b + cx - b = d - cx + cx - b,$$

which, reduced, is

$$ax + cx = d - b.$$

Generally, if a term is to be removed from one member to the other, it must have its sign changed.

44. (2.) When the terms of an equation are fractional, it must be reduced to another whose terms are whole numbers. Take the equation

$$\frac{2x}{3} - \frac{3}{4} = 11 + \frac{x}{5}.$$

Reducing all these fractions to a common denominator by the ordinary method, we have

$$\frac{40x}{60} - \frac{45}{60} = 11 + \frac{12x}{60};$$

and because (43) the two members of an equation may be multiplied by the same number, let us multiply this by 60, which is equivalent to suppressing the denominator 60 in the fractional terms, and multiplying the whole term by 60. The result is

$$40x - 45 = 660 + 12x.$$

In performing this operation, we may arrive at the reduced equation directly, without writing down the common denominator, if we take care to multiply every whole term by that denominator.

Take the equation

$$\frac{5x}{12} - \frac{4x}{3} - 13 = \frac{7}{8} - \frac{13}{6}x.$$

The denominators have evidently common factors, and the least common multiple of the denominators is 24. It is then to this denominator that the fractions must be reduced.

Make this reduction, and omit the common denominator 24, and the result is

$$10x - 32x - 312 = 21 - 52x.$$

(The whole term, -13 , being multiplied by 24.)

Hence this general rule may be deduced; *To clear an equation of fractions, begin by finding the least common multiple of all the denominators.* (This number is the product of all the denominators, if they have no common factor.) *Then multiply every whole term by this common multiple, and every fractional term by the quotient of this common multiple divided by the denominator of that term, and omit that denominator.*

To apply this rule, take the equation

$$\frac{ax}{b} - \frac{2c^2x}{ab} + 4a = \frac{4bc^2x}{a^3} - \frac{5a^3}{b^2} + \frac{2c^3}{a} - 3b.$$

The least common multiple of all the denominators is a^3b^2 . Multiply each whole term by a^3b^2 , and the numerator of each fractional term by the quotient of a^3b^2 , divided by the denominator of that term, and omit the denominator. The result is

$$a^4bx - 2a^2bc^2x + 4a^4b^2 = 4b^3c^2x - 5a^6 + 2a^2b^2c^3 - 3a^3b^3.$$

45. To solve the equation $4x - 3 = 2x + 5$. It becomes by transposition $4x - 2x = 5 + 3$, which reduced, is $2x = 8$.

Divide both members by 2, and we have $x = \frac{8}{2} = 4$.

If 4 be substituted instead of x in the equation, it becomes

$$4 \times 4 - 3 = 2 \times 4 + 5, \text{ or } 13 = 13.$$

Take the equation of article 44

$$\frac{5x}{12} - \frac{4x}{3} - 13 = \frac{7}{8} - \frac{13}{6}x.$$

This, when cleared of fractions, is

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$$10x - 32x - 312 = 21 - 52x.$$

By transposition, it becomes

$$10x - 32x + 52x = 21 + 312;$$

and by reduction,

$$30x = 333, \text{ or } x = \frac{333}{30} = \frac{111}{10},$$

which may be verified by substituting for x this value in the original equation.

Let $(3a - x)(a - b) + 2ax = 4b(x + a).$

The multiplications here indicated must be performed, in order to reduce the two members to simple polynomials, so as to disengage the unknown quantity x . It then becomes

$$3a^2 - ax - 3ab + bx + 2ax = 4bx + 4ab;$$

which, when transposed and reduced, becomes

$$ax - 3bx = 7ab - 3a^2.$$

Now, $ax - 3bx$ is the same thing as $(a - 3b)x$. Therefore,

$$(a - 3b)x = 7ab - 3a^2.$$

Divide both members by $3ab$,

and we have
$$x = \frac{7ab - 3a^2}{a - 3b}.$$

To solve an equation of the first degree, however complicated it may be;

(1.) *Begin by clearing the equation of fractions, if there are any, and performing all the algebraical operations which present themselves on both members of the equation; thus an equation is obtained, the members of which are whole polynomials.*

(2.) *Transpose to one member (the first is generally chosen) all the terms into which the unknown quantity enters, and to the other member all the known terms.*

(3.) *Reduce to one all the terms which contain x , if the equation is numerical; and if it is algebraical, one product composed of two factors must be formed of all these terms, one of which is x , and the other the aggregate of the quantities by which it is multiplied, with their respective signs.*

(4.) *Divide both sides by the number or polynomial by which x is multiplied, and perform the division, if possible.*

This rule can be applied, in all its parts, to the following example;

$$\frac{(a+b)(x-b)}{a-b} - 3a = \frac{4ab - b^2}{a+b} - 2x + \frac{a^2 - bx}{b}.$$

This, cleared of fractions, is

$$\begin{aligned} & b(a+b)^2(x-b) - 3ab(a^2 - b^2) \\ &= b(a-b)(4ab - b^2) - 2b(a^2 - b^2)x + (a^2 - b^2)(a^2 - bx); \end{aligned}$$

when the multiplications are performed,

$$\begin{aligned} & a^2bx + 2ab^2x + b^3x - a^2b^2 - 2ab^3 - b^4 - 3a^3b + 3ab^3 \\ &= 4a^2b^2 - ab^3 - 4ab^3 + b^4 - 2a^2bx + 2b^2x + a^4 - a^2b^2 - a^2bx + b^3x; \end{aligned}$$

which becomes, by transposition and reduction,

$$4a^2bx + 2ab^2x - 2b^3x = 4a^2b^2 - 6ab^3 + 2b^4 + 3a^3b + a^4;$$

collect into one all the terms which contain x , and

$$b(4a^2 + 2ab - 2b^2)x = 4a^2b^2 - 6ab^3 + 2b^4 + 3a^3b + a^4,$$

$$x = \frac{a^4 + 3a^3b + 4a^2b^2 - 6ab^3 + 2b^4}{b(4a^2 + 2ab - 2b^2)},$$

which cannot be reduced to a simple polynomial.

46. In an equation such as $3x - 2 = 4x - 7$, the transposition of the unknown terms to the first member, and the known ones to the second, gives

$$3x - 4x = 2 - 7, \text{ or } -x = -5.$$

To explain this result, observe, that the order of the transposition may be inverted, that is, the unknown terms may be put in the second member of the equation, which will give

$$7 - 2 = 4x - 3x, \text{ or } 5 = x, \text{ or } x = 5;$$

that is, whenever the result is such as $-x = -5$, the signs of the two members must be changed. This is evidently the same thing as transposing the unknown terms into the second member, and the known ones into the first member, and then writing the second first.

We now proceed to the resolution of problems.

47. We have already said, that the first part of the solution of an algebraical problem cannot be reduced to any fixed rule. Sometimes the enunciation of the problem furnishes an equation immediately; sometimes it is necessary to discover from the enunciation conditions from which an equation may be formed; and frequently it is not the conditions of the enunciation which must be translated into algebraical language, but certain conditions derived from them. The first of these are called *explicit conditions*, and those which

are derived from the enunciation *implicit conditions*. Nevertheless, the extended application of the following rule will almost always lead to an equation. *Consider the problem as solved, and having represented the known quantities either by letters or numbers, and the unknown quantity by a letter, write down in algebraical language all the reasonings and operations which would have been necessary to verify the value of the unknown quantity had it been given.*

Problem 1. *To find a number, the half, third, and fourth of which, added to 25, give 448.*

Let x be the number sought; then $\frac{x}{2}$, $\frac{x}{3}$, and $\frac{x}{4}$, are the half, third, and fourth of that number. From the enunciation, these three, added to 45, gives 448, that is,

$$\frac{x}{2} + \frac{x}{3} + \frac{x}{4} + 45 = 448.$$

Transpose 45, and

$$\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 403.$$

Cleared of fractions, this becomes

$$6x + 4x + 3x = 4836, \text{ or } 13x = 4836;$$

whence

$$x = \frac{4836}{13} = 372.$$

$$\text{and } \frac{372}{2} + \frac{372}{3} + \frac{372}{4} + 45 = 186 + 124 + 93 + 45 = 448.$$

This question is one of that species which, in arithmetic, is solved by the *rule of false position*; we see with what ease it is solved by algebra.

Problem 2. *A workman is engaged for 48 days, and receives for each day that he works 24 cents and his food. Each day that he does not work, his food costs him 12 cents; and at the end of the 48 days he receives 504 cents. How many days did he work, and how many was he idle?*

If we knew these two numbers, and multiplied them respectively by 24 and 12, and subtracted the second product from the first, the remainder would be 504. This must be expressed algebraically.

Let x be the number of days during which he works; $48 - x$ is the number of those during which he is idle, $24x$ is the number of cents which he gains, and $12(48 - x)$ that which he loses.

Consequently

$$24x - 12(48 - x) = 504,$$

or

$$24x - 576 + 12x = 504;$$

therefore

$$36x = 1080,$$

and

$$x = \frac{1080}{36} = 30;$$

whence

$$48 - x = 18.$$

Therefore, he works 30 days, and rests 18. For 30 days of work he receives 30×24 , or 720 cents; in 18 days of rest he loses 18×12 , or 216 cents. Now, $720 - 216 = 504$.

This problem may be generalized by calling n the whole number of days of labor and idleness, a the sum he receives for each working day, b that which he loses for each day of idleness, and c his total gain. Let x be the number of working days, and then $n - x$ is the number of days of idleness. Then ax and $b(n - x)$ represent respectively the sums gained and lost.

The equation of the problem is

$$ax - b(n - x) = c,$$

whence

$$ax - bn + bx = c,$$

$$(a + b)x = c + bn,$$

and

$$x = \frac{c + bn}{a + b}.$$

Therefore

$$n - x = n - \frac{c + bn}{a + b} = \frac{an + bn - c - bn}{a + b} = \frac{an - c}{a + b}.$$

Problem 3. *A fox is pursued by a greyhound, and is 60 of her own leaps before him. The fox makes 9 leaps while the greyhound makes 6, but the latter in 3 leaps goes as far as the former in 7, How many leaps does the greyhound take before he catches the fox?*

It is evident that the distance the dog has to run, consists of the 60 leaps which the fox is in advance of him, added to the distance the latter runs, from the moment when the former begins the chase. If then these two distances can be expressed in terms of one unknown quantity, the equation of the problem is easily stated.

Let x be the number of leaps, made by the dog. Since the fox makes 9 leaps, while the dog makes 6, it follows that the fox makes $\frac{9}{6}$, or $\frac{3}{2}$ leaps, while the dog makes one, and therefore $\frac{3x}{2}$ is the number of leaps made by the fox while the dog makes x . It might

be imagined, that to obtain the equation it would be sufficient to put x and $60 + \frac{3x}{2}$ equal to each other; but, by so doing, we should commit a manifest error; for the leaps of the dog exceed those of the fox; and we should thus put equal to each other two heterogeneous numbers, that is, two numbers referred to different units. We must then express the leaps of the fox *in terms* of those of the dog, or the converse. Now 3 leaps of the dog being equivalent to 7 of the fox, one leap of the dog is equivalent to $\frac{7}{3}$ leaps of the fox, and therefore x leaps of the dog are equivalent to $\frac{7x}{3}$ leaps of the fox.

$$\text{Therefore, } \frac{7x}{3} = 60 + \frac{3}{2}x;$$

$$\text{or } 14x = 360 + 9x,$$

$$\text{or } 5x = 360, \text{ and } x = 72.$$

Therefore the dog makes 72 leaps, and the fox makes $72 \times \frac{3}{2}$, or 108 leaps.

Verification.

The 72 leaps of the dog are equivalent to $72 \times \frac{7}{3}$, or 168 of the fox, and $168 = 60 + 108$.

The two following problems are well worthy the attention of learners, as an exercise in algebraic calculation.

48. Problem 4. *A father, who has three children, leaves his property to them by will, in the following way; to the first he bequeaths a sum a , together with the n th part of what remains; to the second a sum $2a$, with the n th part of what remains after the first part and $2a$ has been deducted; to the third, a sum $3a$, with the n th part of what remains after the two first parts and $3a$ has been deducted. The property is thus entirely divided; what is its amount?*

Call this property x . If by the aid of this quantity, we can form the algebraical expressions of the three parts, we must subtract their sum from x , and the remainder, put equal to nothing, gives the equation of the problem.

Since x denotes the property of the father, and $x - a$ the remainder after a has been taken away, the share of the first is

$$a + \frac{x - a}{n}, \text{ or } \frac{an + x - a}{n}.$$

To form the second part, we must subtract from x this first part and $2a$, which gives

$$x - 2a - \frac{an + x - a}{n};$$

or, when reduced,

$$\frac{nx - 3an - x + a}{n}.$$

Now, the second part is $2a$ added to the n th part of this remainder; in other words, it is

$$2a + \frac{nx - 3an - x + a}{n^2},$$

or

$$\frac{2an^2 + nx - 3an - x + a}{n^2}.$$

If we take from x the two first parts and $3a$, the remainder is

$$x - 3a - \frac{an + x - a}{n} - \frac{2an^2 + nx - 3an - x + a}{n^2},$$

which, reduced to the same denominator and simplified, is

$$\frac{n^2x - 6an^2 - 2nx + 4an + x - a}{n^2}.$$

The third part therefore is,

$$3a + \frac{n^2x - 6an^2 - 2nx + 4an + x - a}{n^3},$$

or

$$\frac{3an^3 + n^2x - 6an^2 - 2nx + 4an + x - a}{n^3}.$$

But according to the enunciation of the question, the property of the father is thus entirely used up. Therefore the difference between x and the sum of the three parts must be equal to zero; that is,

$$\left. \begin{aligned} x - \frac{an + x - a}{n} - \frac{2an^2 + nx - 3an - x + a}{n^2} \\ - \frac{3an^3 + n^2x - 6an^2 - 2nx + 4an + x - a}{n^3} \end{aligned} \right\} = 0;$$

which, the denominators being cancelled and the whole reduced, becomes

$$n^3x - 6an^3 - 3n^2x + 10an^2 + 3nx - 5an - x + a = 0;$$

whence

$$x = \frac{6an^3 - 10an^2 + 5an - a}{n^3 - 3n^2 + 3n - 1} = \frac{a(6n^3 - 10n^2 + 5n - 1)}{n^3 - 3n^2 + 3n - 1}.$$

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We might obtain an equation and a result more simply by observing that, since the portion of the third child is $3a$, added to the n th part of the remainder, and the property is then all disposed of, the portion of the third is $3a$, and the remainder spoken of is nothing.

The expression for this remainder is

$$\frac{n^2 x - 6an^2 - 2nx + 4an + x - a}{n^2},$$

which, put equal to nothing, gives

$$n^2 x - 6an^2 - 2nx + 4an + x - a = 0,$$

or
$$x = \frac{6an^2 - 4an + a}{n^2 - 2n + 1} = \frac{a(6n^2 - 4n + 1)}{n^2 - 2n + 1}.$$

To prove the *numerical identity* of this expression with the former one, it is sufficient to show that the second may be derived from the first, by suppressing a factor common to both its terms. Now if to the two polynomials

$$a(6n^2 - 10n^2 + 5n - 1),$$

and

$$(n^3 - 3n^2 + 3n - 1),$$

be applied the rule for finding the greatest common divisor (41), we shall see that $n - 1$ is their greatest common divisor, and by dividing both terms of the first expression by $n - 1$, we obtain the second.

This problem will show the learner the importance of finding out, in the enunciation of a question, all the circumstances which facilitate the formation of an equation, the neglect of which may lead to solutions much more complicated than the question actually requires.

The conditions from which the expressions of the three parts were formed, are the *explicit* conditions of the proposed problem; the condition from which the most simple equation was derived is an *implicit* condition, which a little attention shows to be contained in the enunciation of the question.

To obtain the values of the three parts, it is sufficient to put for x , its value in the expressions obtained above.

$$\text{Let } a = 10000, n = 5.$$

Then

$$x = \frac{10000(6 \times 25 - 4 \times 5 + 1)}{25 - 2 \times 5 + 1} = \frac{10000 \times 131}{16} = \frac{1310000}{16} = 81875$$

Bour. Alg.

The first child must have $10000 + \frac{81875 - 10000}{5}$, or 24375.

There remains then for the other two $81875 - 24375$, or 57500.

The second has $20000 + \frac{57500 - 20000}{5}$, or 27500.

There remains then $57500 - 27500$, or 30000 for the third share.

Now 30000 is triple of 10000, which verifies the solution.

A less direct, but more simple and elegant solution of this problem, may be given. It is founded on the condition, that the subtraction of $3a$ and of the two first parts from the whole, leaves no remainder.

Let r, r', r'' , be the three remainders mentioned in the enunciation; the expression for the three parts will be

$$a + \frac{r}{n}, 2a + \frac{r'}{n}, 3a + \frac{r''}{n}.$$

Now (1.) by the question $r'' = 0$. Thus the third part is $3a$.

(2.) What is left, after the share of the second and first have been deducted, is $r - \frac{r'}{n}$, or $\frac{(n-1)r'}{n}$. And this remainder is the share of the third; therefore

$$\frac{(n-1)r'}{n} = 3a; \text{ whence } r' = \frac{3an}{n-1}.$$

Then the share of the second is

$$2a + \frac{1}{n} \cdot \frac{3an}{n-1} \text{ or } 2a + \frac{3a}{n-1}, \text{ that is } \frac{2an+a}{n-1}.$$

The remainder, after the share of the first has been taken away, is $r - \frac{r}{n}$, or $\frac{(n-1)r}{n}$.

Moreover this must be the sum of the second and third shares; hence

$$\frac{n-1}{n} r = 3a + \frac{2an+a}{n-1} = \frac{5an-2a}{n-1}.$$

$$\text{Therefore } r = \frac{5an-2a}{n-1} \times \frac{n}{n-1} = \frac{5an^2-2an}{(n-1)^2}.$$

And accordingly the first part is $a + \frac{1}{n} \cdot \frac{5an^2-2an}{(n-1)^2}$

$$= a + \frac{5an-2a}{(n-1)^2} = a + \frac{5an-2a}{n^2-2n+1} = \frac{an^2+3an-a}{n^2-2n+1}.$$

The whole property is, therefore,

$$3a + \frac{2an+a}{n-1} + \frac{an^2+3an-a}{(n-1)^2},$$

or
$$\frac{3a(n^2-2n+1) + (2an+a)(n-1) + an^2+3an-a}{n^2-2n+1}$$

$$= \frac{6an^2-4an+a}{n^2-2n+1} = \frac{a(6n^2-4n+1)}{n^2-2n+1},$$

which is the result obtained above.

This solution is more complete than the preceding, since the whole property and the three shares are found at the same time.

49. Problem 5. *A father leaves in his will, to his eldest child, a sum a , and also the n th part of the remainder of his property; to the second, a sum $2a$, with the n th part of what remains, when the first share and $2a$ have been deducted; to the third, a sum $3a$, with the n th part of the new remainder; and so on. Moreover, the shares of all the sons are equal. What is the property of the father, the share of each of the children, and the number of children?*

This problem is remarkable, since it contains more conditions than are necessary to determine the value of the unknown quantities.

Let x be the property of the father; $x-a$ is what remains when a is taken away. Therefore the share of the eldest is

$$a + \frac{x-a}{n}, \text{ or } \frac{na+x-a}{n}.$$

The subtraction of this part and of $2a$ from the whole, leaves

$$x-2a - \frac{an+x-a}{n}, \text{ or } \frac{nx-3an-x+a}{n},$$

of which the n th part is $\frac{nx-3an-x+a}{n^2}$.

Then the share of the second is

$$2a + \frac{nx-3an-x+a}{n^2}, \text{ or } \frac{2an^2+nx-3an-x+a}{n^2}.$$

In the same way the other shares may be found; but as they must all be equal, the equation of the problem may be formed by simply putting the two first shares equal to each other, which gives

$$\frac{an+x-a}{n} = \frac{2an^2+nx-3an-x+a}{n^2};$$

whence

$$x = an^2 - 2an + a.$$

Substitute this value of x in the first share, which then becomes

$$\frac{an + an^2 - 2an + a - a}{n}, \text{ or } \frac{an^2 - an}{n}, \text{ or } an - a,$$

or $a(n - 1);$

and as all the shares must be equal, dividing the whole property by the share of the first we shall have the number of children; this number is $\frac{an^2 - 2an + a}{an - a}$, or $n - 1$.

The whole property is, therefore, $a(n - 1)^2$.

The share of each son is $a(n - 1)$.

The number of children is $n - 1$.

It remains for us to determine whether the other conditions of the problem are satisfied; that is, whether, if the second have $2a$ with the n th part of the remainder, the third $3a$, with the n th part of the remainder, the share of each is $a(n - 1)$.

The difference between the whole property and the first part is $a(n - 1)^2 - a(n - 1)$, and the share of the second is

$$2a + \frac{a(n - 1)^2 - a(n - 1) - 2a}{n},$$

or $\frac{2a(n - 1) + a(n - 1)^2 - a(n - 1)}{n},$

or $\frac{a(n - 1) + a^2(n - 1)^2}{n}, \text{ or } \frac{a(n - 1)(1 + n - 1)}{n}$

or $a(n - 1)$.

In like manner, the difference between the whole and the two first shares is $a(n - 1)^2 - 2a(n - 1)$; and the share of the third is $3a + \frac{a(n - 1)^2 - 2a(n - 1) - 3a}{n}$, which is evidently

$$\frac{a(n - 1) + a(n - 1)^2}{n}, \text{ or } a(n - 1).$$

The fourth share is

$$4a + \frac{a(n - 1)^2 - 3a(n - 1) - 4a}{n},$$

or $\frac{a(n - 1) + a(n - 1)^2}{n};$

and so on. Therefore all the conditions contained in the enunciation are fulfilled.

Of Equations and Problems of the First Degree with Two or more Unknown Quantities.

50. Although several of the problems which we have solved contained more than one unknown quantity, we have employed only one symbol of an unknown quantity in their solution. This was because the conditions of the enunciation enabled us easily to express the other unknown quantities, by means of one of them ; but this cannot be done in all problems which contain more than one unknown quantity.

To know how such problems are to be solved, take first one of those which have been already solved by means of one unknown quantity.

To find two numbers, whose sum is a , and whose difference is b (4).

Call these numbers x and y ; then

$$x + y = a,$$

$$x - y = b.$$

Now, it is an axiom, that if to two equal numbers A and B be added two other equal numbers C and D , the results $A + C$ and $B + D$ are equal ; that is, if $A = B$ and $C = D$, $A + B = C + D$.

Also, if from two equal numbers be subtracted two equal numbers, the remainders are equal ; that is, if $A = B$ and $C = D$, then $A - C = B - D$.

Apply this principle to the equations of the proposed problem ; we shall have,

by adding $2x = a + b,$

and by subtracting $2y = a - b.$

We obtain from the first, $x = \frac{a + b}{2} ;$

and from the second, $y = \frac{a - b}{2}.$

$$\begin{aligned} \text{And } \frac{a + b}{2} + \frac{a - b}{2} &= \frac{2a}{2} = a \text{ and } \frac{a + b}{2} - \frac{a - b}{2} \\ &= \frac{2b}{2} = b. \end{aligned}$$

Take the problem of the labourer (47), considering only the general enunciation.

Let x be the number of working days, and y those of idleness ;
 ax and by are the sums respectively which he receives for his
 work, and loses by his idleness.

The equations are

$$\begin{aligned}x + y &= n, \\ax - by &= c.\end{aligned}$$

Now we have already seen that two equal quantities, multiplied
 by the same quantity, are still equal ; if therefore we multiply both
 members of the first equation by b , the coefficient of y in the
 second, we shall have

$$bx + by = bn,$$

which, added to

$$ax - by = c,$$

gives

$$bx + ax = bn + c,$$

and

$$x = \frac{bn + c}{a + b}.$$

Multiply the first by a , the coefficient of x in the second, and we
 have

$$ax + ay = an.$$

Subtract the second from this, and there remains

$$ay + by = an - c, \text{ or } y = \frac{an - c}{a + b}.$$

The introduction of a symbol to represent each of the unknown
 quantities, has the advantage of finding either of the numbers
 sought independently of the other.

Elimination.

51. Given the two equations,

$$\begin{aligned}5x + 7y &= 43, \\11x + 9y &= 69,\end{aligned}$$

which may be regarded as the algebraical translation of the enun-
 ciation of a problem containing two unknown quantities.

If one of the unknown quantities had the same coefficient in both
 equations, simple subtraction would give another equation contain-
 ing only one unknown quantity, from which the value of that un-
 known quantity might be found.

Multiply both members of the first equation by 9, the coefficient of y in the second, and both members of the second equation by 7, the coefficient of y in the first. This gives

$$45x + 63y = 387,$$

$$77x + 63y = 483,$$

which may be substituted for the two first equations, and in which y has the same coefficient.

Subtract the first from the second, and we have

$$32x = 96, \text{ or } x = 3.$$

In like manner, multiply the first equation by 11, the coefficient of x in the second, and the second by 5, the coefficient of x in the first. This gives

$$55x + 77y = 473,$$

$$55x + 45y = 345.$$

Subtract the second of these from the first, and we obtain

$$32y = 128, \text{ or } y = 4.$$

Then $x = 3$, and $y = 4$, are the two values of x and y , which verify the enunciation of the question. For

$$5 \cdot 3 + 7 \cdot 4 = 15 + 28 = 43,$$

and

$$11 \cdot 3 + 9 \cdot 4 = 33 + 36 = 69.$$

The operation by which the values of the unknown quantities have been found, is known by the name of *elimination*, and consists in making one of the unknown quantities disappear by means of certain transformations of the proposed equation.

The preceding method has a close analogy with that for the reduction of fractions to a common denominator; and, like this last operation, is susceptible of some simplifications.

Take the equations

$$8x - 21y = 33,$$

$$6x + 35y = 177.$$

To render the coefficients of x and y equal, observe that 21 and 35 have a common factor 7; it is sufficient, therefore, to multiply the first equation by 5, and the second by 3, which gives

$$40x - 105y = 165,$$

$$18x + 105y = 531.$$

Add these equations, and we obtain

$$58x = 696, \text{ or } x = 12.$$

In like manner, the two coefficients of x contain a common factor 2; it is sufficient, therefore, to multiply the first equation by 3 and the second by 4, which gives

$$24x - 63y = 99,$$

$$24x + 140y = 708.$$

Subtract the first from the second, and we have

$$203y = 609, \text{ or } y = 3.$$

N. B. It is important to find out whether the coefficients have any common factors, as the calculations are then much more simple.

As a third example, take the equations

$$\frac{2x}{3} - 4 + \frac{y}{2} + x = 8 - \frac{3y}{4} + \frac{1}{12},$$

$$\frac{y}{6} - \frac{x}{2} + 2 = \frac{1}{6} - 2x + 6.$$

They must first be cleared of fractions by the rule (44), and thus the two following equations are obtained,

$$8x - 48 + 6y + 12x = 96 - 9y + 1,$$

$$y - 3x + 12 = 1 - 12x + 36,$$

which, reduced, become

$$20x + 15y = 145,$$

$$9x + y = 25,$$

or

$$4x + 3y = 29,$$

$$9x + y = 25.$$

Multiply the second equation by 3, and subtract the first from the result. This gives

$$23x = 46, \text{ or } x = 2.$$

Now, $y = 25 - 9x = 25 - 9 \cdot 2 = 7.$

52. We now proceed to the solution of three equations containing three unknown quantities.

Let the equations be

$$5x - 6y + 4z = 15,$$

$$7x + 4y - 3z = 19,$$

$$2x + y + 6z = 46,$$

To eliminate z by means of the two first equations, multiply the first by 3 and the second by 4, and add the two results; as the coefficients of z have contrary signs, we obtain

$$43x - 2y = 121.$$

Multiply the second equation by 2, one of the factors of the coefficient of z in the third, and add the result to the third, which gives

$$16x + 9y = 84.$$

The question is then reduced to finding the values of x and y , which satisfy these new equations. Multiply the first by 9 and the second by 2; adding the results, we have

$$419x = 1257, \text{ or } x = 3.$$

We might now determine y as we have determined x ; but it may be done more simply by putting for x its value in the second equation, which gives

$$48 + 9y = 84, \text{ or } y = \frac{84 - 48}{9} = 4.$$

Put for x and y their values in the first of the three proposed equations, and it becomes

$$15 - 24 + 4z = 15, \text{ or } z = \frac{24}{4} = 6.$$

Let there be m equations, with m unknown quantities. To find the value of the unknown quantities, combine successively one of the equations with each of the $m - 1$ others, so as to eliminate the same unknown quantity from each; thus are obtained $m - 1$ equations, containing $m - 1$ unknown quantities, on which perform the process which has already been performed on the first equations; that is, eliminate another unknown quantity, by combining one of the new equations with the $m - 2$ others, which leaves $m - 2$ equations, containing $m - 2$ unknown quantities. Continue this series of operations until at last there remains one equation, containing only one unknown quantity, from which find the value of that unknown quantity. Then, by going back through the series of equations which have been obtained, the value of the other unknown quantities may be successively determined.

53. The mode of elimination, here given, is known by the name of the *method by addition and subtraction*; because the unknown quantities disappear by addition and subtraction, when the equations have been so transformed that one unknown quantity has the same coefficient in two of them.

There are two other methods of elimination commonly used. The first, called the *method by substitution*, consists in finding from one equation the value of one unknown quantity, as if the others had been determined, and substituting this value in the other equations, which produces new equations, containing one unknown quantity less than the original ones, on which the same operation is repeated. The second, called the *method by comparison*, consists in finding

the value of the same unknown quantity from all the equations, and putting these values equal to one another. This gives rise to new equations, containing one unknown quantity less than the original ones, on which the process is repeated.

But these two methods have an inconvenience from which the method by addition and subtraction is free. The new equations contain denominators which must be made to disappear. The method by substitution may be employed with advantage whenever the coefficient of an unknown quantity is unity in one of the equations, because then this inconvenience does not exist. Sometimes it will be necessary to employ it. But generally, the method by addition and subtraction is preferable; it has, besides, this advantage, that if the coefficients be not too great, the addition or subtraction may be performed at the same time with the multiplication, which makes the coefficients equal.

54. It often happens that the given equations do not each contain all the unknown quantities. In this case, with a little skill, the eliminations may be shortened.

Take the equations

$$2x - 3y + 2z = 13 \dots\dots (1),$$

$$4u - 2x = 30 \dots\dots (2),$$

$$4y + 2z = 14 \dots\dots (3),$$

$$5y + 3u = 32 \dots\dots (4).$$

It will be perceived immediately that the elimination of z between equations (1) and (3) gives an equation in x and y ; also, that if u be eliminated between (2) and (4), a second equation in x and y is obtained; these two unknown quantities can then be easily determined.

First, the elimination of z between (1) and (3) gives

$$7y - 2x = 1;$$

that of u between (2) and (4) gives $20y + 6x = 38$.

Multiply the first of these by 3, and add; which gives

$$41y = 41, \text{ or } y = 1.$$

Substitute this value in the equation $7y - 2x = 1$; we find $x = 3$.

Substitute the value of x in (2), which becomes

$$4u - 6 = 30, \text{ or } u = 9.$$

The substitution of the value of y in (3) gives $z = 5$.

The student may take as an exercise the following equations ;

$$\left. \begin{array}{l} 7x - 2z + 3u = 17 \\ 4y - 2z + t = 11 \\ 5y - 3x - 2u = 8 \\ 4y - 3u + 2t = 9 \\ 3z + 8u = 33 \end{array} \right\} \begin{array}{l} \text{which give} \\ x = 2, y = 4, z = 3, u = 3, t = 1. \end{array}$$

55. In all that has gone before, we have supposed the number of equations equal to the number of unknown quantities. This must be the case in every problem where there are unknown quantities, in order that it may be determinate, that is, that it may not admit an infinite number of solutions.

Suppose, for example, that a problem, with two unknown quantities x and y , leads to the single equation $5x - 3y = 12$;

whence
$$x = \frac{12 + 3y}{5}.$$

If for y be put successively

$$1, 2, 3, 4, 5, 6 \dots\dots,$$

the values of x are

$$3, \frac{18}{5}, \frac{21}{5}, \frac{24}{5}, \frac{27}{5}, 6 \dots\dots;$$

and each of the systems of values

$$x = 3 \qquad y = 1,$$

$$x = \frac{18}{5} \qquad y = 2,$$

$$x = \frac{21}{5} \qquad y = 3,$$

will satisfy the equation.

If there are two equations with three unknown quantities, one of the unknown quantities may be first eliminated by means of the given equations ; and thus an equation is obtained, which, as it contains two unknown quantities, may be satisfied by an infinite number of values of these unknown quantities, whence an infinite number of values of the third unknown quantity may be obtained. Then, in all cases, in order that a problem may be determinate, it is necessary that its enunciation contain at least as many different conditions as there are unknown quantities, and that each condition be expressible by an equation.

56. We go on to the resolution of problems containing two or more unknown quantities.

Problem 6. *A person possesses a capital of \$30000, on which he gains a certain rate of interest ; but he owes \$20000, for which he pays interest at another rate. The interest which he receives is greater than that which he pays by \$800. A second person has \$35000, on which he gains the second rate of interest ; but he owes \$24000, for which he pays the first rate of interest. The sum which he receives is greater than that which he pays by \$310. What are the two rates of interest ?*

Let x and y be the two rates of interest. The interest of \$30000 at the rate of x per cent. is x taken as many times as there are hundreds in 30000 ; that is $300x$. Also, the interest of \$20000 at y per cent. is $200y$. And, from the question, the difference of these two is \$800.

Then the first equation of the problem is

$$300x - 200y = 800.$$

The second condition in the question, translated into algebraical language in the same way, is

$$350y - 240x = 310.$$

As both numbers of the first equation are divisible by 100 and of the second by 10, they are equivalent to

$$3x - 2y = 8,$$

$$35y - 24x = 31.$$

To eliminate x , multiply the first equation by 8, and add the result to the second, which gives $19y = 95$, or $y = 5$. Substitute this in the first equation, which becomes

$$3x - 10 = 8, \text{ or } x = 6.$$

Therefore the first rate of interest is 6 and the second 5 per cent.

30000 at 6 per cent. gives yearly 300×6 , or 1800 ; 20000 at 5 per cent. gives 200×5 , or 1000 ; and

$$1800 - 1000 = 800.$$

In the same way the second condition might be verified.

Problem 7. *There are three ingots of different metals mixed together. A pound of the first contains 7 ounces of silver, 3 ounces of copper, and 6 of pewter ; a pound of the second contains 12 ounces of silver, 3 of copper, and 1 of pewter ; that of the third, 4 ounces of silver, 7 of copper, and 5 of pewter. How much of each of the three ingots must be taken to form a fourth, which shall*

contain in a pound, 8 ounces of silver, $3\frac{1}{2}$ of copper, and $4\frac{1}{2}$ of pewter?

Let x , y , and z be the number of ounces which must be taken from the three ingots to form a pound of the fourth. Since in the first there are 7 ounces of silver in a pound of 16 ounces, it follows that in one ounce there is $\frac{7}{16}$ of an ounce of silver, and in x ounces there must be $\frac{7x}{16}$ of silver. In the same way we might show that

$\frac{12y}{16}$ and $\frac{4z}{16}$ are the quantities of silver taken from the second and third ingots to form a pound of the fourth; but, by the question, this contains 8 ounces of silver;

therefore
$$\frac{7x}{16} + \frac{12y}{16} + \frac{4z}{16} = 8,$$

or
$$7x + 12y + 4z = 128.$$

By reasoning in the same way on the quantities of copper and pewter, contained in the fourth, we have

$$3x + 3y + 7z = 60,$$

and
$$6x + y + 5z = 68.$$

As in these equations the coefficients of y are the most simple, it will be convenient to eliminate y first.

Subtract the first equation from the second, multiplied by 4, and we obtain

$$5x + 24z = 112.$$

Subtract the second equation from the third, multiplied by 3, and the result is

$$15x + 8z = 144.$$

Subtract the first of these two from the second, multiplied by 3, and we have

$$40x = 320, \text{ or } x = 8.$$

Substitute this value of x in the second of the above, and it becomes

$$120 + 8z = 144, \text{ or } z = 3.$$

The values of z and of x , substituted in the third of the original equations, give

$$48 + y + 15 = 68, \text{ or } y = 5.$$

Thus, to form a pound of the fourth ingot, we must take 8 ounces of the first, 5 of the second, and 3 of the third. If in 16

ounces of the first, there are 7 of silver, in 8 ounces there must be $\frac{7 \times 8}{16}$ of the same. In like manner, $\frac{12 \times 5}{16}$ and $\frac{4 \times 3}{16}$ are the quantities of silver contained in 5 ounces of the second ingot and 3 of the third. Now,

$$\frac{7 \times 8}{16} + \frac{12 \times 5}{16} + \frac{4 \times 3}{16} = \frac{128}{16} = 8,$$

and the fourth ingot contains 8 ounces in the pound of silver, as the question requires. In the same way the conditions arising from the quantity of copper and pewter may be verified.

57. The following problems are intended as an exercise for the student.

(1.) *One workman can finish a things in b days; a second can finish c things in d days; and a third, e things in f days. How long will they be, all working together, in finishing g things?*

$$\text{Answer, } \frac{b d f g}{a d f + b c f + b d e} \text{ days.}$$

Application.

$$a = 27$$

$$b = 4,$$

$$c = 35$$

$$d = 6,$$

$$e = 40$$

$$f = 12,$$

$$g = 191.$$

Answer, 12 days.

(2.) *If 32 pounds of sea water contain 1 pound of salt, how much fresh water must be added to these 32 pounds, in order that in 32 pounds of the new mixture the quantity of salt may be reduced to 2 ounces, or $\frac{1}{4}$ of a pound?*

Answer, 224 pounds.

(3.) *How many times do the hands of a clock meet between noon and midnight, and at what hours?*

Ans. 11 times; at $1^h 5' \frac{5}{11}$, $2^h 10' \frac{10}{11}$, $3^h 16' \frac{4}{11}$, &c.

(4.) *A number is composed of three digits whose sum is 11, the units digit is double of that of the hundreds, and 297 added to this number gives the number inverted. What is the number?*

Answer, 326.

(5.) *A person who has \$100000 puts part to interest at 5 and part at 4 per cent.; his income is \$1640. What are the two parts?*

Answer, \$64000 and \$36000.

(6.) *A person has a certain capital, which brings a certain interest. A second person, who has \$10000 more than the first and gets 1 per cent. more, has a greater income by \$800. A third*

who has \$15000 more than the first, and gets 2 per cent. more, has a greater income by \$1500. What are the three capitals, and the rates of interest?

Ans. The capitals are \$30000, \$40000, and \$45000.

The rates are 4, 5, and 6 per cent.

Problems which give rise to Negative Results. Theory of Negative Quantities.

58. The use of algebraical signs in the resolution of problems often leads to results which at first view are embarrassing; on reflection, however, it will appear, not only that they are capable of explanation, but that by their means the language of algebra may be still further generalized.

Let us take the following problem. *To find a number which, added to a number b , gives for their sum a number a .*

Let x be the number required, then evidently

$$b + x = a, \text{ or } x = a - b.$$

This expression or formula gives the value of x , in all particular cases of the problem.

For example, let

$$a = 47, b = 29;$$

then

$$x = 47 - 29 = 18.$$

Let now $a = 24$, $b = 31$; then $x = 24 - 31$. Since

$$31 = 24 + 7,$$

this expression may be put under the form $x = 24 - 24 - 7$, or, by reduction, $x = -7$. This value of x is what is called a *negative solution*. What is the meaning of it?

If we return to the statement of the problem, we see that it is impossible that 31 added to any number can give 24, a number less than 31. Therefore no number can solve the question in this case. Nevertheless, if, in the equation of the problem, $31 + x = 24$, we put, instead of the term $+x$, the negative value -7 , it becomes $31 - 7 = 24$, a true equation, which amounts to saying that 31 diminished by 7 is 24.

The negative solution $x = -7$ indicates the impossibility of solving the problem in the sense in which it was proposed; but if we consider the solution independently of its sign, that is, $x = 7$, we may see that it is the solution of the following problem, *To find*

a number which, subtracted from 31, gives 24; which only differs from the first, viz. To find a number which, added to 31, gives 24, in this, that the words *added to* are supplied by the words *subtracted from*.

The new question, when solved directly, gives the equation

$$31 - x = 24;$$

whence

$$31 - 24 = x \text{ or } x = 7.$$

Let us take the following problem. *A father is a years, his son b years old. In how many years will the son's age be one-fourth of his father's?*

Let x be the number of years, then $a + x$ and $b + x$ represent the ages of the father and son at the end of this number of years; then the equation is,

$$b + x = \frac{a + x}{4}, \quad \text{or } x = \frac{a - 4b}{3}.$$

Let $a = 54$, and $b = 9$; then

$$x = \frac{54 - 36}{3} = \frac{18}{3} = 6.$$

The father being 54 years old, and the son 9, in six years the father will be 60, and the son 15; now, 15 is the fourth of 60; therefore 6 is the answer to the problem.

Let us suppose $a = 45$, $b = 15$;

$$\text{then } x = \frac{45 - 60}{3}.$$

This expression may be reduced to $x = -5$ by the ordinary rules. How is this negative result $x = -5$ to be explained?

If we return to the equation of the problem, we shall find that in this case it becomes

$$15 + x = \frac{45 + x}{4}.$$

This contains a manifest contradiction; for the second member is $\frac{45}{4} + \frac{x}{4}$; and each of these two parts is less than the corresponding part of the other member. But if we substitute -5 for $+x$, it

$$\text{becomes } 15 - 5 = \frac{45 - 5}{4}, \text{ or } 10 = \frac{40}{4};$$

an exact equation, which indicates that if, instead of adding to the ages of the two, we take away 5 years, the age of the son will be one-fourth of that of the father. Thus the solution which has been

found, considered independently of its sign, is the solution of the following problem. *A father is 45 years old, and his son 15; when was the age of the son one-fourth of that of the father?*

The equation of this new problem is

$$15 - x = \frac{45 - x}{4};$$

whence $60 - 4x = 45 - x$, and $x = 5$.

The least consideration of the problem will show that as the ratio of the ages of the two is $\frac{15}{45}$, or $\frac{1}{3}$, the age of the son cannot become one-fourth of that of the father, but has been so already; because, as has been proved (6), by adding to both terms of a fraction the same number, the fraction is increased in value. On the contrary, it is diminished in value by the subtraction of the same quantity from both terms.

59. We are led by these analogies to the establishment of the following principle.

(1.) *When the unknown quantity is found to have a negative value, it is indicative of some incorrectness in the manner of stating the question, or, at least, in the equation which is the algebraical translation of it.* (See the Remark at the end of this article.)

(2.) *This value, independently of its sign, may be regarded as the solution of a problem, which differs from the proposed problem in this, that certain quantities, which were additive in the first, are subtractive in the second, and the reverse.*

Demonstration. The first part of this principle may be easily demonstrated. The finding a negative value for x must arise from our being led, by the nature of the equation, to subtract the greater of two numbers from the less, *which is impossible*.

Thus the values $x = -7$, $x = -5$, (58) arose from the equations

$$x = 24 - 31, \quad x = \frac{45 - 60}{3}.$$

Now, if no absolute number,* when substituted for x , can verify the equation of the problem, after the transformations (43, 45) have been made, it follows that the original equation itself cannot be verified by the substitution of any absolute number for x ; for the correctness of the transformations has been shown for all equations which are capable of being verified.

* An absolute number is one considered without reference to its sign, as in arithmetic.

Sometimes the impossibility of solving the problem in the way in which it has been put, is evident on the mere inspection either of the enunciation or the equation; the two preceding problems are examples of this. Sometimes it is difficult to discover this impossibility; but it is always made evident in the course of the solution.

We now go on to the second part of the principle.

Observe, first, that if, in the equation, $-x$ be substituted for $+x$, all the terms containing x , which were additive, are now subtractive, and the reverse. If there be, for example, the term $+ax$, when $-x$ is put instead of x , it becomes $+a \times -x$, or $-ax$. In like manner, if we have the term $-bx$, it becomes $-b \times -x$ or $+bx$. If this new equation be translated into ordinary language, a new enunciation is obtained, which differs from the first only in having some quantities subtractive which before were additive, and the reverse.

It remains to show that the substitution of $-x$ in the place of x gives $x = p$, the former result being $x = -p$. (p is here considered as an absolute number).

Now, whatever the original equation of the problem may be, we can always reduce it *by known transformations* to the form $ax = +b$ (a and b being absolute numbers).

From this equation,

$$x = \frac{-b}{a}, \text{ or } x = -\frac{b}{a}, \text{ or } x = -p,$$

if p be put for the absolute number $\frac{b}{a}$. But if $-x$ be put instead of x in the original equation, the same transformations will reduce the new equation to the form

$$-ax = -b,$$

from which
$$x = \frac{-b}{-a} = \frac{b}{a} = p,$$

which was to be demonstrated.

It will hence be seen how we are to interpret negative results. When the sign is taken away, they may be regarded as the solutions, not of the questions proposed, but of questions of the same nature, certain conditions of which have been modified; and the surest method of obtaining the new question is to change x into $-x$ in the equation of the problem, and translate the result into ordinary language.

60. *Remark.* The principle which has been established is rigorously true for the equations only, and not always for the enunciations of problems; that is, the enunciation of a problem may be correct, although the resolution of the equation gives a negative value. The cause of this is, that the algebraist, in the application of his methods to the resolution of a problem, is apt frequently to interpret certain conditions in a sense exactly opposite to that in which they ought to be taken; in this case, the negative solution corrects the effects of the wrong view which he has taken of these conditions. Thus, the equation is false, although the problem is capable of being resolved; and it is only when the equation is a faithful translation of the enunciation and of the meaning of all its conditions, that the principle is applicable to the enunciation. We shall see examples of this in what follows; but it is mostly in the application of Algebra to Geometry that the principle is applicable, not to the enunciations, but to the equations.

61. In the preceding demonstration, we have been led to multiply $+a$ by $-x$, to divide $-b$ by $+a$, and $-b$ by $-a$, and the results were obtained by applying to simple quantities the rule of the signs established for the multiplication and division of polynomials. It may appear at first view necessary to demonstrate these rules with reference to insulated simple quantities, and this is what most authors have attempted to do. But the demonstrations which they have given have only the appearance of rigour, and leave much uncertainty in the mind. We say then, that the rule for the signs, established for polynomial quantities, is extended to simple quantities, in order to interpret the peculiar results to which algebraical operations lead. Those who do not admit this extension deprive themselves of one of the principal advantages of algebraical language, which consists in comprehending, under one formula, the solutions of several questions of the same nature, whose enunciations differ only in the way of stating certain conditions, that is, in certain quantities which are additive in the first, being subtractive in the second, and the reverse.

The extension to simple quantities of the rules established for polynomials, may appear desirable from the following considerations.

The demonstration of article 17 for the multiplication of the binomials $a - b$ and $c - d$, evidently supposes $a > b$ and $c > d$. For, if the contrary be the case, the course of reasoning loses all meaning; nevertheless, having once established the rule for the

signs, we need not call it in question, whatever be the magnitudes of a , b , c , and d .

If this be granted, the product of $a - b$ by c , being $a c - b c$, it follows that the product of a negative quantity $a - b$ (a being $< b$) by a positive quantity c , is negative.

Also, the product of b by $c - d$, being $b c - b d$, it follows that the product of a positive quantity b by a negative expression $c - d$ (c being $< d$), is negative.

Lastly, the product of $a - b$ by $c - d$ being

$$a c - b c - a d + b d,$$

which may be put under the form

$$b d - a d - b c + a c, \text{ or } d(b - a) - c(b - a),$$

if we suppose $d > c$ and $b > a$, or $b - a$ positive, it follows that

$$d(b - a) > c(b - a), \text{ or } d(b - a) - c(b - a)$$

is positive. The product therefore of a negative expression $a - b$ by a negative expression $c - d$ (a being $< b$ and $c < d$) is positive.

It is this which constitutes one of the distinguishing characteristics of Algebra. In arithmetic and geometry, the things reasoned on are real, and such that the mind can form a distinct conception of them; while in algebra, the subjects of the reasoning and of the operations are often *imaginary*, or contain symbols of operations which cannot be performed; but the exactness of the results which are obtained by these means, and which may always be arrived at by processes more rigorous, but much longer, is a sufficient sanction of the methods which we have followed.

62. As the rule for the signs applied to simple quantities is of constant use in algebra, we shall here present it at one view. We shall also see methods of deriving new expressions peculiar to algebraical language.

We begin by addition and subtraction.

To add $+b$ or $-b$ to a quantity expressed by a , we must write the result thus, $a + b$ or $a - b$; that is, we must write the two simple quantities one after the other, with their respective signs. (13.)

The subtraction of $+b$ or $-b$ from a , must be written thus, $a - b$ or $a + b$; that is, change the sign of the simple quantities to be subtracted, and write it with its new sign, after that from which it is to be taken. (14.)

In multiplication and division,

$$\left. \begin{array}{l} +a \times +b \text{ or } -a \times -b \text{ gives the product } +ab \\ -a \times +b \text{ or } +a \times -b \text{ gives the product } -ab \end{array} \right\} (17.)$$

$$\left. \begin{array}{l} \frac{+a}{+b} \text{ or } \frac{-a}{-b} \text{ gives the quotient } +\frac{a}{b} \\ \frac{-a}{+b} \text{ or } \frac{+a}{-b} \text{ gives the quotient } -\frac{a}{b} \end{array} \right\} (25.)$$

These rules give rise to the following important remarks.

(1.) In algebra, the words *add* and *sum* do not always, as in arithmetic, convey the idea of increase; for the result $a - b$, which arises from the addition of $-b$ to a , is, properly speaking, the difference between the number of units in a and that in b ; whence the result is less than a . To distinguish this kind of sum from an arithmetical sum, call it the *algebraical sum*. Thus the polynomial

$$2a^3 - 3a^2b + 3b^2c - 2a^2c$$

is an *algebraical sum*, if we regard it as the result of the union of the simple quantities

$$2a^3, -3a^2b, +3b^2c, \text{ and } -2a^2c,$$

with their respective signs; its *real meaning* is the arithmetical difference between the sum of the units contained in the positive and negative terms.

It follows, that an algebraical sum may, if particular values be substituted for the letters which it contains, be a negative quantity, or one which is preceded by the sign $-$.

(2.) The words *subtract* and *difference* do not always convey the idea of diminution; for the difference between a and $-b$, being $a + b$, is greater than a ; it is called an *algebraical difference*, because it can be put under the form $a - (-b)$.

By means of these denominations, negative values may be regarded as solutions of equations. For example, in the equation $31 + x = 24$; the result, $x = -7$, indicates that we must add -7 to 31 to obtain 24; and, in fact, $31 + (-7)$, or $31 - 7$, is 24.

In like manner, in the equation

$$15 + x = \frac{45 + x}{4},$$

the result, $x = -5$, shows that we must add -5 to the two ages, in order that the age of the son may be one-fourth of that of the father.

For $15 + (-5)$, or $15 - 5 = 10$,
 $45 + (-5)$, or $45 - 5 = 40$.

63. The necessity of employing negative expressions in algebraical calculations, and of proceeding with them as with absolute quantities, leads to two other propositions, which are of frequent use in algebra. Every negative quantity $-a$, is less than 0, and of two negative quantities, the least is that whose numerical value is the greatest.

Thus $-a < 0$, and $-a < -b$, if a be numerically greater than b .

Demonstration. To explain these two propositions, observe, that if from the same number we subtract a series of continually increasing numbers, the remainders will continually diminish. Take any whole number, as 6 for example, and from it subtract successively 1, 2, 3, 4, 5, 6, 7, 8, 9, &c. and we shall have, writing the differences in the same line,

$6-1, 6-2, 6-3, 6-4, 6-5, 6-6, 6-7, 6-8, 6-9, \&c.$

which, reduced, become

5, 4, 3, 2, 1, 0, -1, -2, -3, &c.

Whence it will be seen, that -1 must be regarded as less than nothing, because the last expresses the difference between 6 and itself, whereas -1 expresses the difference between 6 and a greater number.

For the same reason, -1 is greater than -2 , -3 is greater than -12 , although the numerical values of the first expressions are less than those of the last.

Another Demonstration. Since, in order to interpret the peculiar results which the algebraical solution of a problem leads to, we have agreed to consider *negative expressions* as quantities, we ought, in performing with them the same operations as with absolute numbers, to arrive at true results. Now, we may regard it as an axiom, that if a number a is greater than another number b , and the same number d be added to both, the first result $a + d$ is greater than the second $b + d$.

Then, if we admit that $0 > -a$, and $-a > -(a + m)$, (a and m being absolute numbers), if we add $a + m$ to the two members of each, we find $a + m > m$, and $m > 0$, which is true. On the contrary, if we supposed $0 < -a$, and $-a < -(a + m)$, the same reasoning would give us $a + m < m$ and $m < 0$, which is absurd.

The two preceding propositions must be admitted, if we are to treat negative expressions as absolute quantities. These propositions are an algebraical method of speaking, analogous to that which we often make use of in common language. We say often that a man is worse *than nothing*, to express that his debts exceed his property; and, of two persons of equal fortune, each of whom owes more than he possesses, that *the richest is he who owes least*.

Discussion of Problems of the First Degree, containing One or more Unknown Quantities.

64. After a problem has been solved generally, that is, after the given quantities are expressed by letters, we can determine what the values of the unknown quantities are, when particular hypotheses are made respecting the given quantities. The determination of their different values, and the interpretation of the peculiar results at which we arrive, form what is called the *discussion of the problem*.

The discussion of the following question will present nearly all the circumstances which are usual in problems of the first degree.

$\overline{R \quad A \quad B \quad R}$

Problem 14. *Two couriers set off at the same time from two different points, A and B, of the same straight line AR, and in the same direction AB. The one who starts from A goes m, and the other n miles an hour. At what distance from the points A and B will the two couriers meet?*

Solution. Let R be the point at which they meet; let the unknown distances AR and BR , expressed in miles, be called x and y , and let a be the distance AB between them at the time when they start. Then

$$x - y = a \dots \dots (1).$$

Since m and n are the number of miles per hour (or the respective velocities of the two couriers), it follows, that the times of travelling over the spaces x and y , are expressed by $\frac{x}{m}$ and $\frac{y}{n}$; and these times are equal; therefore

$$\frac{x}{m} = \frac{y}{n}, \text{ or } nx - my = 0 \dots \dots (2).$$

The combination of equations (1) and (2), by the known methods of elimination, gives

$$x = \frac{am}{m-n}, \quad y = \frac{an}{m-n},$$

which values are easily verified.

As long as we suppose $m > n$, or $m - n > 0$, or positive, these values are positive, and the problem, as the enunciation stands, admits of a real solution. And it is plain, that if the courier A goes faster than B , he gains on him every instant; the distance which separated them at first diminishes more and more, till at last it is entirely destroyed, and the two are at the same point of the line which they are travelling.

But if we suppose $m < n$, or $m - n < 0$, or negative, the values of x and y are negative, and are

$$x = -\frac{am}{n-m}, \quad y = -\frac{an}{n-m}.$$

In order to interpret these results we observe, that it is impossible for the two couriers to meet in the direction AB ; for as B goes faster than A , the distance which separates them increases every moment. But if, instead of supposing that they proceed in the direction AB , we suppose, on the contrary, that they go in the direction BA , the circumstances of the problem become the same as those of the case where $m > n$; and it is plain that the two couriers will meet in the point R' of BA produced. This is shown also by the principle established in article 59. Change the signs of x and y , and the equations become

$$\left. \begin{aligned} -x + y &= a \\ -\frac{x}{m} &= -\frac{y}{n} \end{aligned} \right\} \text{ or } \left\{ \begin{aligned} y - x &= a \\ \frac{x}{m} &= \frac{y}{n} \end{aligned} \right.$$

which give

$$x = \frac{am}{n-m}, \quad y = \frac{an}{n-m}.$$

These values verify the new enunciation, in which the couriers were supposed to start in the direction BA .

Let $m = n$, or $m - n = 0$, the general values become

$$x = \frac{am}{0}, \quad y = \frac{an}{0}.$$

How are these results to be interpreted?

The enunciation shows, that the solution of this case of the problem is absolutely impossible; that is, in whatever direction the two couriers start, they can never come together; because, being at a certain distance from one another at starting, and going equally fast, they ought to preserve the same distance.

The result $\frac{am}{0}$ may be regarded as a new sign of impossibility.

The equations of the problem become, when $m = n$,

$$\left. \begin{array}{l} x - y = a \\ \frac{x}{m} = \frac{y}{m} \end{array} \right\} \text{ or } \left\{ \begin{array}{l} x - y = a \\ x - y = 0, \end{array} \right.$$

which are evidently inconsistent.

Nevertheless, it is customary to regard such results as

$$x = \frac{am}{0}, \quad y = \frac{an}{0},$$

as a species of value, to which is given the name of infinite. The reason is as follows;

When the difference $m - n$, without being absolutely nothing, is supposed to be very small, the two results

$$\frac{am}{m-n} \text{ and } \frac{an}{m-n}$$

are very great.

For example, let

$$m - n = 0,01, \quad m = 3, \quad \text{whence } n = 3 - 0,01 = 2,99.$$

$$\frac{am}{m-n} = \frac{3a}{0,01} = 300a, \quad \frac{an}{m-n} = 299a.$$

Let $m - n = 0,0001$, $m = 3$, whence $n = 2,9999$; then

$$\frac{am}{m-n} = 30000a, \quad \frac{an}{m-n} = 29999a.$$

In a word, if the difference of the velocities of the couriers be nothing, the two couriers will meet; but the distance between their point of meeting and that of starting, will be greater and greater, the more that difference is diminished. If, then, we suppose that difference less than any given quantity, the distances

$$\frac{am}{m-n} \text{ and } \frac{an}{m-n}$$

will be greater than any given quantity, or infinite. We say then,

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for brevity sake, that when $m - n = 0$, the results become

$$x = \frac{am}{0}, y = \frac{an}{0}, \text{ or infinite.}$$

Since 0 is less than any absolute magnitude, we can take this symbol to represent the last state of any magnitude which may be diminished as much as we please. And since a fraction is greater, according as its numerator is greater compared with its denominator, an expression such as $\frac{A}{0}$ (where A is an absolute magnitude) is well suited to represent an *infinite* quantity, that is, a quantity which is greater than any assignable quantity.

An infinite quantity is also expressed thus, ∞ ; and a quantity less than any given magnitude, or 0, can be expressed by $\frac{A}{\infty}$; for a fraction is the smaller, according as its denominator is greater compared with its numerator. Thus, 0 and $\frac{A}{\infty}$ are synonymous symbols; so also are $\frac{A}{0}$ and ∞ .

We have insisted particularly on these last notions, because there are questions of such a nature that *infinity* may be regarded as a real answer to the question. Frequent examples of this are seen in the application of algebra to geometry.

If to the hypothesis $m = n$, be added this, $a = 0$, the two values become

$$x = \frac{0}{0} \text{ and } y = \frac{0}{0}.$$

What meaning is to be attached to this result?

Returning to the enunciation, it will appear that, as the two couriers go equally fast, and set out from the same point, they will be always together, and consequently they meet at every point of the line which they pass over. Indeed, by this double hypothesis, $m - n, a = 0$, the equations become,

$$\left. \begin{array}{l} x - y = 0 \\ \frac{x}{m} - \frac{y}{m} = 0 \end{array} \right\} \text{ or } \left\{ \begin{array}{l} x - y = 0 \\ x - y = 0, \end{array} \right.$$

which result in the same. Therefore, the question is entirely indeterminate (55), because there is in reality only one equation between two unknown quantities.

The expression $\frac{0}{0}$, in this case, is a symbol that the question thus enunciated is indeterminate (55).

If the two couriers do not go equally fast, that is, if $m >$ or $< n$, while we suppose $a = 0$, the result is $x = 0$, and $y = 0$. As the two set out from the same point with different velocities, they cannot evidently be together except at the point from which they set out.

The preceding hypotheses are the only ones which lead to remarkable results. They serve also to show to beginners in what manner algebra answers all the circumstances contained in the enunciation of a problem.

65. When a problem has been solved generally, we may find, by simple changes of sign in the formulas obtained for the unknown quantities, the solutions of other general problems, the enunciations of which do not differ from those of the proposed problem, excepting that certain quantities which were additive become subtractive, and the reverse.

Let us take, for example, the problem of the workman, which was solved in article 47. If we call c the sum he has to receive, the equations are,

$$x + y = n, \quad ax - by = c,$$

or
$$x = \frac{bn + c}{a + b}, \quad y = \frac{an - c}{a + b}.$$

But, if we suppose that at the day of settling, the workman owes a sum c , instead of having to receive it, the equations are,

$$\left. \begin{array}{l} x + y = n \\ by - ax = c \end{array} \right\} \text{ or } \left\{ \begin{array}{l} x + y = n, \\ ax - by = -c, \end{array} \right.$$

(in which the signs of the second equation are changed).

It is plain that without solving these equations again, we may obtain immediately the corresponding values of x and y by simply changing the sign of c in the preceding values, which gives

$$x = \frac{bn - c}{a + b}; \quad y = \frac{an + c}{a + b}.$$

To prove this rigorously, let us designate $-c$ by d ; the equations become

$$\begin{array}{l} x + y = n, \\ ax - by = d, \end{array}$$

which differ from those of the first enunciation only in having d

instead of $-c$. Then we must have

$$x = \frac{bn + d}{a + b}; y = \frac{an - d}{a + b}.$$

If we put instead of d its value $-c$, the equations become

$$x = \frac{bn + (-c)}{a + b}; y = \frac{an - (-c)}{a + b};$$

or, by applying the rules established in article 62,

$$x = \frac{bn - c}{a + b}; y = \frac{an + c}{a + b}.$$

The results which correspond to the two enunciations may be comprehended in the same formula,

$$x = \frac{bn \pm c}{a + b}, y = \frac{an \mp c}{a + b}.$$

The double sign $\pm$ is called plus or minus; the upper signs refer to the case where the workman has to receive a sum c , and the lower to the case where he has to pay it.

These formulas, moreover, contain the case in which, upon the settlement, the workman is supposed to have nothing either to receive or to pay. It is only necessary to suppose $c = 0$, which gives

$$x = \frac{bn}{a + b}, y = \frac{an}{a + b}.$$

Let there be the two general equations, supposed to arise from the translation of the conditions of a problem into algebraical language,

$$\begin{aligned} ax + by &= c, \\ dx + fy &= g. \end{aligned}$$

By multiplying the first by f , the second by b , and subtracting the second result from the first, we have

$$(af - bd)x = cf - bg, \text{ or } x = \frac{cf - bg}{af - bd}.$$

In like manner, we shall find

$$y = \frac{ag - cd}{af - bd}.$$

This being premised, to pass from these formulas,

(1.) To those corresponding to the equations

$$\begin{aligned} ax - by &= c, \\ dx + fy &= g, \end{aligned}$$

it is sufficient to change b into $-b$, which gives

$$x = \frac{cf + bg}{af + bd}, \quad y = \frac{ag - cd}{af + bd};$$

(2.) To those corresponding to the equations

$$ax - by = c,$$

$$dx - fy = g,$$

it is sufficient to change b into $-b$, and f into $-f$, which gives

$$x = \frac{-cf + bg}{-af + bd} = \frac{bg - cf}{bd - af}, \quad y = \frac{ag - cd}{bd - af}.$$

The demonstration is exactly the same as that of the preceding example.

General Investigation of Equations and Problems of the First Degree.

66. In order to generalize the investigation of problems of the first degree with one or more unknown quantities, we propose to establish formulas which will represent the values of the unknown quantities for any set of equations, containing an equal number of unknown quantities.

In the first place, every equation of the first degree with one unknown quantity may, by means of the usual transformations, be reduced to the form $ax = b$, a designating the algebraic sum of the quantities by which the unknown quantity is multiplied, and b the algebraic sum of all the known terms.

We may evidently deduce from this equation $x = \frac{b}{a}$.

In the second place, we observe that every equation of the first degree with two unknown quantities, may be represented by

$$ax + by = c.$$

For, if the proposed equation contains denominators, we may make them disappear (44); by uniting all the terms containing x and all the terms containing y in the first member, then transposing all the known terms to the second member, we designate the *algebraic sum* of the first by ax , the *algebraic sum* of the second by by , and the *algebraic sum* of the last by c ; a , b , c are then whole quantities affected with either sign.

Let the two proposed equations be $\begin{cases} ax + by = c, \\ a'x + b'y = c'. \end{cases}$

We obtain by multiplying the first equation by b' , the second by b , and subtracting one from the other,

$$(a b' - b a') x = c b' - b c',$$

whence

$$x = \frac{c b' - b c'}{a b' - b a'};$$

we find, in like manner,

$$y = \frac{a c' - c a'}{a b' - b a'}.$$

Let there now be the three equations

$$a x + b y + c z = d \dots\dots\dots (1),$$

$$a' x + b' y + c' z = d' \dots\dots\dots (2),$$

$$a'' x + b'' y + c'' z = d'' \dots\dots\dots (3).$$

In order to eliminate z , we multiply the first equation by c' , the second by c , and subtract the second from the first.

It becomes

$$(a c' - c a') x + (b c' - c b') y = d c' - c d' \dots\dots (4).$$

Combining, in like manner, the second equation with the third, we find

$$(a' c'' - c' a'') x + (b' c'' - c' b'') y = d' c'' - c' d'' \dots\dots (5).$$

In order to eliminate y , we must multiply equation (4) by $b' c'' - c' b''$, and equation (5) by $b c' - c b'$, and then subtract, which gives

$$[(a c' - c a') (b' c'' - c' b'') - (a' c'' - c' a'') (b c' - c b')] x = (d c' - c d') (b' c'' - c' b'') - (c' d'' - d' c'') (b c' - c b');$$

by performing the operations indicated, reducing and dividing the two members by c' , we have

$$(a b' c'' - a c' b'' + c a' b'' - b a' c'' + b c' a'' - c b' a'') x = d b' c'' - d c' b'' + c d' b'' - b d' c'' + b c' d'' - c b' d''.$$

Then finally,

$$x = \frac{d b' c'' - d c' b'' + c d' b'' - b d' c'' + b c' d'' - c b' d''}{a b' c'' - a c' b'' + c a' b'' - b a' c'' + b c' a'' - c b' a''}.$$

By performing analogous operations for eliminating x and z , and afterwards x and y , we shall find for y and z ,

$$y = \frac{a d' c'' - a c' d'' + c a' d'' - d a' c'' + d c' a'' - c d' a''}{a b' c'' - a c' b'' + c a' b'' - b a' c'' + b c' a'' - c b' a''};$$

$$z = \frac{a b' d'' - a d' b'' + d a' b'' - b a' d'' + b d' a'' - d b' a''}{a b' c'' - a c' b'' + c a' b'' - b a' c'' + b c' a'' - c b' a''}.$$

As beginners may not be much used to making all possible abbreviations in the calculation, we will here give a method of passing from the value of x to that of y and z , without being obliged to go over all the preceding operations from the beginning.

We observe that the set of equations (1), (2), and (3), will remain the same if we substitute for x, a, a', a'' , the quantities y, b, b', b'' , and the reverse; then, if in the expression which gives the value of x , we change x into y , then a, a', a'' , which are the coefficients of x , into b, b', b'' , which are the coefficients of y , and the reverse, we shall obtain a result which will be no other than the value of y .

Performing this change, we have

$$y = \frac{d a' c'' - d c' a'' + c d' a'' - a d' c'' + a c' d'' - c a' d''}{b a' c'' - b c' a'' + c b' a'' - a b' c'' + a c' b'' - c a' b''}$$

or, changing the signs of the numerator and denominator, and writing in each, the last three terms first, and the first three terms last,

$$y = \frac{a d' c'' - a c' d'' + c a' d'' - d a' c'' + d c' a'' - c d' a''}{a b' c'' - a c' b'' + c a' b'' - b a' c'' + b c' a'' - c b' a''}$$

In like manner, we should obtain the value of z by changing x, a, a', a'' into z, c, c', c'' , and the reverse.

We can easily see the method which must be pursued, if we had four equations with four unknown quantities, &c.

N. B. By reflecting a little on the manner in which these formulas have been obtained, we shall easily perceive that for any number whatever of equations, containing an equal number of unknown quantities, $x, y, z \dots$, there can exist, in general, but one set of values of $x, y, z \dots$, which will verify the equations.

First, the proposition is evident for an equation with one unknown quantity $a x = b$. There is only the value $\frac{b}{a}$ which can satisfy it.

Let us consider two equations with two unknown quantities.

After we have multiplied the first equation by the coefficient of y in the second, and the reverse, the result which we obtain, by subtracting one from the other, may be substituted in one of the two proposed equations. Now this result, containing only one unknown quantity, admits for that unknown quantity but one value, which, carried back into one of the proposed equations, will likewise

give but one value for y . The same reasoning will apply to three equations with three unknown quantities.

67. The use of accents, in the notation of coefficients, has led to the observation of a law, according to which we can easily find the preceding formulas, without being obliged to perform the elimination.

Let us consider, first, the example of two equations with two unknown quantities. We have found, for the values,

$$x = \frac{c b' - b c'}{a b' - b a'}, \quad y = \frac{a c' - c a'}{a b' - b a'}.$$

(1.) *To obtain the common denominator to these two values, form with the letters a and b , which designate the coefficients of x and of y , in the first equation, the two arrangements $a b$ and $b a$, then interpose the sign $-$, which gives $a b - b a$; finally, accent in each term, the last letter;*

it becomes $a b' - b a'.$

(2.) *In order to obtain the numerator relative to each unknown quantity, replace, in the denominator, the letter which designates the coefficient of that unknown quantity, by the letter which designates the known quantity, still leaving the accents in the same place. Thus, $a b' - b a'$ is changed into $c b' - b c'$, for the value of x , and into $a c' - c a'$, for the value of y .*

Let us consider the case of three equations with three unknown quantities, a, b, c , designating the coefficients of x, y, z , and d the known quantity.

(1.) *In order to have a common denominator, take the denominator $a b - b a$, which corresponds to the case of two unknown quantities, (the accents being omitted); introduce the letter c into each of the terms $a b$ and $b a$, at the right, in the middle, and at the left; then place between them alternately the signs plus and minus; there results*

$$a b c - a c b + c a b - b a c + b c a - c b a.$$

Afterwards place, in each term, the accent ' over the second letter, and the accent '' over the third letter; and we have for the denominator,

$$a b' c'' - a c' b'' + c a' b'' - b a' c'' + b c' a'' - c b' a''.$$

(2.) *In order to form the numerator of each unknown quantity, replace in the denominator, the letter which designates the coefficient of that unknown quantity, by the letter which designates the known*

quantity, leaving the accents in the same place. Thus for x change a into d ; for y , b into d ; and for z , c into d .

This law, which may be regarded as the result of observation for two or three equations, is capable of being extended to any number of equations; but the demonstration is very complicated, and does not belong to the elements of algebra. The learner may consult, for this purpose, the second part of the Algebra of Garnier, which refers to one of M. Laplace. This demonstration is taken from the Memoirs of the Academy of Sciences, 1772.

68. Let us attend to the use which may be made of these formulas, in particular applications.

Let there be the two equations

$$5x - 7y = 34, \quad 3x - 13y = -6.$$

By comparing them with the general equations,

$$ax + by = c, \quad a'x + b'y = c',$$

we have

$$a = 5, \quad b = -7, \quad c = 34, \quad a' = 3, \quad b' = -13, \quad c' = -6.$$

Let us substitute in the formulas

$$x = \frac{cb' - bc'}{ab' - ba'}, \quad y = \frac{ac' - ca'}{ab' - ba'},$$

in the place of a, b, c, a', b', c' , these values; and we have

$$\begin{aligned} (1.) \quad x &= \frac{34 \times -13 - (-7) \times -6}{5 \times -13 - (-7) \times 3} = \frac{-34 \times 13 - 7 \times 6}{-5 \times 13 + 7 \times 3} \\ &= \frac{-442 - 42}{-65 + 21} = \frac{-484}{-44} = 11; \end{aligned}$$

$$(2.) \quad y = \frac{5 \times -6 - 34 \times 3}{5 \times -13 - (-7) \times 3} = \frac{-30 - 102}{-65 + 21} = \frac{-132}{-44} = 3;$$

and $x = 11, y = 3$, are the values which will satisfy the two proposed equations.

We can immediately assure ourselves of this, by substituting them in the equations. But, in order that the demonstration may be independent of every particular example, we remark, that in order to pass from the formulas relative to the equations

$$ax + by = c$$

and $a'x + b'y = c'$, to those which belong to the equations

$$ax - by = c, \quad \text{and} \quad a'x - b'y = -c',$$

it is sufficient (65) to change b into $-b$, b' into $-b'$, and c' into $-c'$, which gives

$$x = \frac{c \times -b' - (-b) \times -c'}{a \times -b' - (-b) \times a'}, \quad y = \frac{a \times -c' - c \times a'}{a \times -b' - (-b) \times a'};$$

and, in order to deduce from these new general formulas, the values which belong to the particular equations, we must make

$$a = 5, b = 7, c = 34, a' = 3, b' = 13, c' = 6.$$

Then, finally, to obtain the values relative to the proposed equations, it is sufficient to make, in the general formulas before obtained,

$$a = 5, b = -7, c = 34, a' = 3, b' = -13, c' = -6,$$

then to perform the calculations, according to the rules laid down for simple quantities.

The rule consists, in general, in substituting, in the place of the coefficients, a, b, a', b' . . . , their particular values considered with the signs by which they are affected in the particular equations, and in performing all the operations indicated according to the established rules.

These applications justify again the necessity of extending to simple quantities the rules of the signs established for polynomial quantities, since it is the means of rendering general formulas of the first degree, applicable to every particular example.

We proceed to the discussion of these formulas.

69. It results from inspection, that in particular applications, we can obtain four kinds of values for answers to problems of the first degree, that is, *positive values, negative values, values of the form $\frac{A}{0}$, in fine, values of the form $\frac{0}{0}$* . The problem of the couriers has given rise to these four results, which we now propose to explain in a general manner.

First, the *positive values* are usually answers to questions in the sense of their enunciation. However, we observe that for certain problems all the positive values do not satisfy the enunciation. If, for example, the nature of the problem requires that the numbers sought should be whole numbers, and that we should find fractional numbers, the problem cannot be resolved. Sometimes, also, the nature of the problem does not permit that the unknown numbers should exceed the numbers known and given *a priori*, or that they should be less than other numbers. If the values obtained, although positive, do not satisfy that condition which the enunciation requires, but which cannot be expressed by an equation, the problem cannot be resolved. Thus the *positive values of unknown*

quantities are, properly speaking, direct answers to the equations ; and they are solutions of the question, only so far as their nature agrees with that which the enunciation requires. In order to conceive how a number can verify an equation, without verifying the problem of which it is the algebraic translation, it is sufficient to remark, that the *same equation is the algebraic translation of an infinity of problems*, of which some admit all possible absolute numbers for a solution, and others admit only numbers of a certain nature.

70. We already know how to regard *negative values* in problems of one unknown quantity. To complete the view already taken, we proceed to consider the principle of article 59, for a problem having several unknown quantities.

It is evident, in the first place, that if we obtain negative values for some of the unknown quantities, the equations of the problem cannot be satisfied in the sense in which they have been established ; for, if a set of absolute numbers, put for x, y, z , could verify them, the equations which have been deduced by the method of elimination, would themselves exist for that set. So that the equation which contains only one of the unknown quantities, for which we have obtained a negative result, would be verified by an absolute number, which would be contrary to the hypothesis. *We must then rectify the enunciation of the problem, or at least, the equations which are the algebraic translation of it.*

If, in the equations, we change the signs of the unknown quantities, for which we have obtained negative results, the terms affected by these unknown quantities will necessarily change their signs, and the enunciation of the problem will be generally modified so that certain quantities, which were additive, will become subtractive, and the reverse.

Finally, these modifications once made, the new enunciation is verified by the values first obtained for the unknown quantities, their signs being left out of consideration. Let us take, for example, three equations with three unknown quantities,

$$ax + by + cz = d, \quad a'x + b'y + c'z = d', \quad a''x + b''y + c''z = d'',$$

and suppose that these equations have given $x = p, y = -q, z = -r$; let us change, in these equations, y and z into $-y$ and $-z$, or rather into y' and z' (designating for the moment, $-y$ and $-z$ by y' and z').

We have

$$\begin{aligned}ax + by + cz &= d, & a'x + b'y' + c'z' &= d', \\a''x + b''y' + c''z' &= d''.\end{aligned}$$

Now, as these equations differ from the preceding, only by having substituted for y and z , y' and z' , they will necessarily give $x = p$, $y' = -q$, $z' = -r$; whence, putting $-y$ and $-z$ in the place of y' and z' . . . , $x = p$, $-y = -q$, $-z = -r$, or rather, $x = p$, $y = q$, $z = r$; which was to be demonstrated.

Thus the principle of article 59 is true with regard to problems of the first degree having several unknown quantities.

We shall conclude with this observation, that sometimes the enunciation of a problem is not, by its nature, susceptible of any modification; in this case, the negative values are only solutions of modified equations, which may moreover be regarded as the algebraic translation of other problems susceptible of modification.

71. It now remains to interpret such expressions as $\frac{A}{0}$, $\frac{0}{0}$.

Let there be, in the first place, the equation with one unknown quantity, $ax = b$; whence $x = \frac{b}{a}$.

(1.) If, for a particular hypothesis, made with respect to the given quantities of the question, we have $a = 0$, there results $x = \frac{b}{0}$.

Now the equation becomes, in this same case, $0 \times x = b$, and can evidently be satisfied by no determinate number. Let us however remark, that as the equation may also take the form $\frac{b}{x} = 0$, if we put in the place of x numbers continually increasing, $\frac{b}{x}$ will continually diminish, and the equation will approach nearer and nearer to being exact; so that we can take for x a value sufficiently great to make $\frac{b}{x}$ less than any assignable quantity.

It is for this reason that Algebraists are accustomed to say, that in this case infinity satisfies the equation; and there are some questions for which this kind of result forms a true solution; at any rate it is certain that the equation cannot admit of a solution in a finite number, and this is all which we wish to prove.

(2.) If we have, at the same time, $a = 0$, $b = 0$, the value of x takes the form $x = \frac{0}{0}$.

Now, the equation becomes, in this case, $0 \times x = 0$, and every finite number, positive or negative, will satisfy this equation.

So that the equation (or the problem of which it is the algebraic translation) is indeterminate.

72. It is important here to remark upon the expression $\frac{0}{0}$, that it does not always announce an *indeterminateness*; but rather the *existence of a factor common* to the two terms of the fraction, which factor becomes zero by the effect of a particular hypothesis.

Let us suppose, for example, that we have found for the result of the solution of a problem,

$$x = \frac{a^3 - b^3}{a^2 - b^2}.$$

If, in this formula, we make $a = b$, there results $x = \frac{0}{0}$.

But we remark, that $a^3 - b^3$ can (31) be put under the form $(a - b)(a^2 + ab + b^2)$, and $a^2 - b^2$ is equal to $(a - b)(a + b)$; so that the value of x is reduced to

$$x = \frac{(a - b)(a^2 + ab + b^2)}{(a - b)(a + b)}.$$

Now, if before making the hypothesis $a = b$, we begin by suppressing the common factor $a - b$, the value of x becomes

$$x = \frac{a^2 + ab + b^2}{a + b},$$

an expression which is reduced to

$$x = \frac{3a^2}{2a}, \text{ or } x = \frac{3a}{2},$$

on the hypothesis of $a = b$.

Let there be, for a second example, the expression

$$\frac{a^2 - b^2}{(a - b)^2} = \frac{(a + b)(a - b)}{(a - b)(a - b)}.$$

By making $a = b$, we find, for the value of x , $x = \frac{0}{0}$, on account of the existence of the common factor $a - b$; but if we first suppress this factor, it becomes $x = \frac{a + b}{a - b}$, an expression which is reduced to $x = \frac{2a}{0}$, when we make $a = b$.

We hence conclude, that the symbol $\frac{0}{0}$ is sometimes, in Algebra, characteristic of the existence of a factor common to the two terms of the fraction which is reduced to this form. So that, before pronouncing upon the true value of the fraction, we must ascertain if these two terms do not contain a common factor. If there is not one, we conclude, that the equation is really *indeterminate*. If there is one, we suppress it, then make again the particular hypothesis, which gives the true value of the fraction, which may still be presented under three forms, $\frac{A}{B}, \frac{A}{0}, \frac{0}{0}$; in which case, the equation is determinate, impossible in a finite number, or indeterminate.

This observation is very useful in the discussion of problems.

73. Let us return to our subject, and consider now two equations with two unknown quantities,

$$ax + by = c, \quad a'x + b'y = c'.$$

We have found (66), for the values of x and y ,

$$x = \frac{cb' - bc'}{ab' - ba'}, \quad y = \frac{ac' - ca'}{ab' - ba'}.$$

Let us suppose that we have $ab' - ba' = 0$, the numerators $cb' - bc'$, $ac' - ca'$, being of different values from 0; the values are reduced to

$$x = \frac{A}{0}, \quad y = \frac{B}{0}.$$

In order to interpret these results, let us remark, that from the equation $ab' - ba' = 0$, we obtain $a' = \frac{ab'}{b}$, whence, substituting this value in the equation $a'x + b'y = c'$, we have

$$\frac{ab'}{b}x + b'y = c';$$

or, making the denominator to disappear, and dividing by b' ,

$$ax + by = \frac{bc'}{b'},$$

an equation whose first member is identical with that of the first

$$ax + by = c,$$

while the second member is essentially different; for, from the inequality $cb' > bc'$ we deduce $c > \frac{bc'}{b'}$.

We see then that the two proposed equations cannot be satisfied simultaneously by any set of finite values of x and y .

If we have, at the same time,

$$a b' - b a' = 0, \quad c b' - b c' = 0,$$

the value of x is reduced to $x = \frac{0}{0}$, a value which needs to be interpreted.

The two proposed equations may, in consequence of the relation $a b' - b a' = 0$, be put under the form

$$a x + b y = c, \quad a x + b y = \frac{b c'}{b'},$$

equations which necessarily return into each other; for from the relation $c b' - b c' = 0$, we deduce $c = \frac{b c'}{b'}$.

So that in order to resolve the problem, we have but one equation with two unknown quantities. Then, *the question is indeterminate.*

Since the relation, $a b' - b a' = 0$, gives $b' = \frac{b a'}{a}$, whence, substituting in the relation

$$b c' - b c' = 0, \quad \frac{c b a'}{a} - b c' = 0,$$

or reducing, $c a' - a c' = 0$, we may conclude that *if the value of x is of the form $\frac{0}{0}$, the value of y is GENERALLY of the same form, and the reverse.*

We say *generally*, for, if we had at the same time $b = 0, b' = 0$, the two expressions, $a b' - b a', c b' - b c'$, would be necessarily zero, without there hence resulting any determinate value for $a c' - c a'$.

In this particular case, the two values of x and y are reduced to

$$x = \frac{0}{0}, \text{ and } y = \frac{a c' - c a'}{0}, \text{ or } \frac{A}{0}.$$

Reciprocally, if we had at the same time $a = 0, a' = 0$, there would hence result $x = \frac{c b' - b c'}{0}$ or $\frac{A}{0}$, and $y = \frac{0}{0}$.

But this particular case is hardly admissible, since the proposed equations would be reduced to two equations with one unknown quantity,

$$\text{namely, } \left. \begin{array}{l} a x = c \\ a' x = c' \end{array} \right\} \text{ supposing } b = 0, b' = 0;$$

$$\text{and } \left. \begin{array}{l} b y = c \\ b' y = c' \end{array} \right\} \text{ supposing } a = 0, a' = 0;$$

while we are considering here the case of two equations with two unknown quantities.

74. It would be difficult to apply the preceding method to the case where we had more than two equations; but we can supply the deficiency by the following reasonings.

In order to present the subject clearly, let us consider four equations (1), (2), (3), (4), containing the four unknown quantities x, y, z, u .

Let us designate by $\frac{A}{D}$ the value of x , at which we have arrived by the assistance of elimination, and suppose that, for a certain hypothesis made upon the given terms, we have $D = 0$, A being any quantity whatever or equal to 0; in the first case, *the proposed equation cannot be satisfied by finite values*; and in the second, *they are indeterminate*, or susceptible of being verified by an infinity of sets of values of x, y, z, u .

In fact, it results from the method of elimination, that the set of equations (1), (2), (3), and (4), may be replaced by four other equations, of which one is $Dx = A$; the second is an equation in terms of x and y ; the third, an equation in terms of x, y, z , and the fourth, one of the proposed equations, the equation (1), for example.

This being premised, (1.) if the value of x is reduced to the form $\frac{A}{0}$, since the equation in x becomes then $0 \times x = A$, and as it is besides a consequence of the simultaneous existence of the proposed equations, these equations must be *impossible* in finite numbers, since the equation $0 \times x = A$ cannot be satisfied by any finite number.

(2.) If the value of x is reduced to the form $\frac{0}{0}$ (without the existence of any common factor between the two terms of its expression), the equation $Dx = A$ becomes $0 \times x = 0$, and may be satisfied by an infinity of values of x . Substituting each of these values in the equation in x and y , of which we have spoken above, we shall obtain an infinity of values corresponding to y ; substituting all these sets of values of x and y , in the equation in x, y, z , we shall find an infinity of values for z ; finally, let us substitute all these sets of values of x, y , and z , in the equation (1), and there will hence result an infinity of values for u ; and all these sets, thus obtained, will necessarily satisfy the four proposed equations.

75. The first part of this proposition is subject to no restriction ; whenever we find for the value of one of the unknown quantities, a result of the form $\frac{A}{0}$, it is a certain sign that the equations are *impossible* in *finite* numbers, at least for that unknown quantity.

As to the second part, it admits of numerous modifications ; that is, we may obtain for one or several of the unknown quantities, results of the form $\frac{0}{0}$, without for this reason concluding that the equations are indeterminate. Sometimes even, one of the unknown quantities having a value of the form $\frac{0}{0}$, we obtain the others under the form $\frac{A}{0}$.

The following sets furnish a proof of this ;

$$\begin{aligned} (1.) \text{ set of equations } & \begin{cases} ax + by + cz = d, \\ ax + by + cz = d', \\ ax + by + cz = d'', \end{cases} \\ (2.) \text{ set of equations } & \begin{cases} ax + by + cz = d, \\ a'x + by + cz = d', \\ a''x + by + cz = d'', \end{cases} \\ (3.) \text{ set of equations } & \begin{cases} x + y + mz = p, \\ x + y + mz = q, \\ x + y + nz = r. \end{cases} \end{aligned}$$

If we apply the general formulas of article 66, to the first set, we find

$$x = \frac{0}{0}, \quad y = \frac{0}{0}, \quad z = \frac{0}{0};$$

and still, by considering this set, we easily discover that it cannot exist in finite numbers (the first members remaining the same), except we have $d = d' = d''$. It is true, that from the moment when this relation exists, the set becomes *indeterminate*, since it is reduced then to a single equation with three unknown quantities ; but it is not less certain that, in its actual state, there is an incompatibility between the equations.

The first of the formulas of article 66, applied to the second set, gives $x = \frac{0}{0}$, a result that requires to be explained.

In order that the second set may exist, (the first members remaining the same,) we must have

$$d - ax = d' - a'x, \text{ and } d - ax = d'' - a''x.$$

These equations give successively

$$x = \frac{d' - d}{a' - a} \text{ and } x = \frac{d'' - d}{a'' - a};$$

but these two values of x ought to be equal, so that we have the equality of condition

$$\frac{d' - d}{a' - a} = \frac{d'' - d}{a'' - a}.$$

So long as this relation among the quantities a, d, a', d', a'', d'' , is not satisfied, the second set will be impossible, although we have obtained values of the form $\frac{0}{0}$ for each of the unknown quantities.

If this relation is satisfied, the value of x becomes determinate and equal to

$$\frac{d' - d}{a' - a} \text{ or } \frac{d'' - d}{a'' - a}.$$

But the values of y and z are indeterminate, since we have only one equation between two unknown quantities.

The formulas of article 66 applied to the third set, give

$$x = \frac{(m-n)(p-q)}{0}, \quad y = \frac{(n-m)(p-q)}{0}, \quad z = \frac{0}{0};$$

that is, two values are of the form $\frac{A}{0}$, and the other of the form $\frac{0}{0}$.

In this example, in order that the two first equations may be possible simultaneously, (the first members remaining the same,) we must have $p = q$; in which case, the values of x and y are reduced also to the form $\frac{0}{0}$.

By admitting this condition we say, the value of z , for which we have found $\frac{0}{0}$, becomes determinate, and the two others are indeterminate.

For, the set is then reduced to the two equations

$$x + y + mz = p, \quad x + y + nz = r,$$

which, subtracted one from the other, give

$$(m-n)z = p-r; \text{ whence } z = \frac{p-r}{m-n}.$$

Carrying back this value into the two equations, we obtain the simple equation

$$x + y = \frac{m r - p n}{m - n}.$$

The cases which we have just examined are sufficient to convince us that in the applications of general formulas to particular sets,

(1.) The values of unknown quantities may be presented, some under the form $\frac{A}{0}$, others under the form $\frac{0}{0}$.

(2.) That the symbol $\frac{0}{0}$, obtained for each of them, is not necessarily characteristic of an indeterminateness in the equations.

(3.) That the symbol $\frac{A}{0}$ is always a character of *impossibility*; but the symbol $\frac{0}{0}$ is sometimes a character of *indeterminateness*, and sometimes a character of impossibility. It sometimes also announces the presence of a common factor (72).

That we may know what to hold as to the true import of $\frac{0}{0}$, we cannot do better than return, in each particular set, to the equations of that set, and seek again directly the values of the unknown quantities, with the help of these equations.

76. Let there be, for the first example, the set of equations,

$$\begin{aligned} x + 9y + 6z &= 16, \\ 2x + 3y + 2z &= 7, \\ 3x + 6y + 4z &= 13. \end{aligned}$$

By applying the general formulas, we find for the three unknown quantities

$$x = \frac{0}{0}, \quad y = \frac{0}{0}, \quad z = \frac{0}{0};$$

but if we operate directly upon the equations, we find, by multiplying the second by 3, and subtracting the first from the second,

$$5x = 21 - 16 = 5;$$

whence

$$x = 1.$$

Substituting this value in the three equations, we obtain
 for the 1st, $9y + 6z = 15$, or $3y + 2z = 5$;
 for the 2d, $\dots\dots\dots 3y + 2z = 5$;
 and for the 3d, $6y + 4z = 10$, or $3y + 2z = 5$.

The value of x is then *determinate* and equal to 1; as to the values of y and z , they are *indeterminate*, since we have only one equation with these two unknown quantities.

The proposed set enters into the second of the preceding article; since, if we divide the first equation by 3, and the third by 2, the coefficients of y and z become the same.

It is, moreover, formed in this manner. After having placed arbitrarily the first members of these three equations, so that the coefficients of y and z , multiplied crosswise in the equations, considered two and two, may always form equal products, we have also taken arbitrarily the second members of the first and second equation, which gave

$$\left. \begin{array}{l} x + 9y + 6z = 16 \\ 2x + 3y + 2z = 7 \\ 3x + 6y + 4z = k \end{array} \right\} \text{whence} \left\{ \begin{array}{l} \frac{1}{3}x + 3y + 2z = \frac{16}{3}, \\ 2x + 3y + 2z = 7, \\ \frac{3}{2}x + 3y + 2z = \frac{k}{2}. \end{array} \right.$$

Then we determined $\frac{k}{2}$ or d'' , from the relation

$$\frac{d'' - d}{a'' - a} = \frac{d' - d}{a' - a}$$

of article 75, by putting

$$a = \frac{1}{3}, d = \frac{16}{3}, a' = 2, d' = 7, a'' = \frac{3}{2},$$

which gave $d'' = \frac{13}{2}$; whence $2d''$ or $k = 13$.

Let there be the new set

$$11x - 8y + 6z = 49,$$

$$5x - 12y + 9z = 16,$$

$$4x - 20y + 15z = 15,$$

which enters also into the second of article 75, but does not satisfy the relation

$$\frac{d'' - d}{a'' - a} = \frac{d' - d}{a' - a}.$$

By applying the formulas, we should find

$$x = \frac{0}{0}, y = \frac{A}{0}, z = \frac{B}{0}.$$

But let us operate directly on the equations.

Multiplying the first by 3, and the second by 2, then subtracting, we obtain $23x = 115$; whence $x = 5$. Substituting this value in the three equations, we find

$$\left. \begin{array}{l} 8y - 6z = 6 \\ 12y - 9z = 9 \\ 20y - 15z = 5 \end{array} \right\} \text{or rather } \left\{ \begin{array}{l} 4y - 3z = 3, \\ 4y - 3z = 3, \\ 4y - 3z = 1. \end{array} \right.$$

The last two equations are evidently impossible simultaneously; and, if we apply to them the formulas relative to two unknown quantities, we shall obtain $y = \frac{A}{0}, z = \frac{B}{0}$.

So that of the three values $\frac{0}{0}$, obtained above, for x, y, z , the first has a determinate signification $x = 5$, and the two others are *infinite*.

77. In fine, while we operate directly upon particular examples, we have other characteristics of *impossibility* or *indeterminateness*.

In the first set, treated of in article 76, if we consider the two equations

$$3y + 2z = 5, \quad 3y + 2z = 5,$$

at which we arrived by the elimination of x ; and, in order to obtain y or z subtract these two equations, the one from the other, we have $0 = 0$.

In the same manner, in the second set, if we subtract one from the other, the two equations

$$4y - 3z = 3, \quad 4y - 3z = 1,$$

we have $0 = 2$, which is absurd.

The results, $0 = 0$, and $0 = A$, are the true characteristics of *indeterminateness*, or *simultaneous impossibility* of the equations.

78. We shall terminate the discussion of equations of the first degree by the examination of a particular case. It is that in which, in the general equations, all the known quantities which are in the second member, are simultaneously zero.

In this case, it evidently follows from the law of the formation of the numerators, for the general values of unknown quantities, (67), that these numerators are all destroyed at once, that is, that we

have $A = 0, B = 0$. Since, moreover, there exists no particular relation between the coefficients a, b, c, a', b', c' , of the unknown quantities, D , which results from a certain combination of these coefficients, is generally different from 0. So that we have, for the values of the unknown quantities, $x = 0, y = 0, z = 0$. These values evidently verify the proposed equations.

If, however, besides the hypothesis that the known quantities of the second member, are simultaneously zero, we have still between the coefficients of the unknown quantities, the relation $D = 0$, the general values are reduced to the form $x = \frac{0}{0}, y = \frac{0}{0}, \&c.$

Now we say, that in this case, the equations are indeterminate, but the ratios of the unknown quantities are constant numbers, which may be obtained by the help of the proposed equations.

Let there be the three equations

$$ax + by + cz = 0, a'x + b'y + c'z = 0, a''x + b''y + c''z = 0,$$

in which we suppose (67) that we have D or

$$a b' c'' - a' c' b'' + c a' b'' - b a' c'' + b' c' a'' - c b' a'' = 0.$$

They may be put under the form

$$a \frac{x}{z} + b \frac{y}{z} + c = 0, a' \frac{x}{z} + b' \frac{y}{z} + c' = 0, a'' \frac{x}{z} + b'' \frac{y}{z} + c'' = 0.$$

Now, we draw from the first two, by treating $\frac{x}{z}$ and $\frac{y}{z}$ as two unknown quantities.

$$\frac{x}{z} = \frac{b c' - c b'}{a b' - b a'}, \frac{y}{z} = \frac{c a' - a c'}{a b' - b a'},$$

Whence we see that by giving to z values entirely arbitrary, the values of x and y will be obtained by the help of these two proportions whose second results are constant and equal to known quantities.

But it remains to know if these values satisfy the third equation. We find by substituting them in this equation,

$$a'' \times \frac{b c' - c b'}{a b' - b a'} + b'' \times \frac{c a' - a c'}{a b' - b a'} + c'' = 0,$$

or reducing and writing the terms in a proper order,

$$a b' c'' - a' c' b'' + c a' b'' - b a' c'' + b' c' a'' - c b' a'' = 0,$$

a condition which, by hypothesis, is satisfied.

79. This leads naturally to the examination of a circumstance of which the second problem of the will, resolved (49), has given an example ; it is that where the enunciation of the question leads to a number of really different equations, greater than that of the unknown quantities to be determined.

Let us suppose, for more generality, that the question contains n unknown quantities, and gives rise to m different equations, m being $> n$. *We must first combine with them a number n of the proposed equations, in order to obtain the values in the n unknown quantities ; then substitute these values $m - n$ remaining equations, which give rise to as many relations between the given terms ; and these last relations ought to be verified, in order that the problem may be possible in the form in which it has been enunciated. The $m - n$ relations thus obtained, are called equations of condition.*

80. *Recapitulation of the preceding discussion.* It results from this discussion, (1.) that a set of equations of the first degree with an equal number of unknown quantities, can in general be satisfied but in one way (66) ;

(2.) That every *positive value*, found for an unknown quantity, answers directly to the equations of the problem, without corresponding always to the enunciation (69) ;

(3.) That every *negative value* answers only indirectly to the enunciation or to the equations which are its algebraic translation, but always corresponds to the equations considered in a sense purely algebraic (59 and 70) ;

(4.) That every expression of the form $\frac{A}{0}$, found for one or several unknown quantities, indicates an incompatibility in the proposed set of equations, at least, in *finite* numbers, for all unknown quantities (71, 73, and 74) ;

(5.) That every expression of the form $\frac{0}{0}$ indicates, either an indeterminateness, or an incompatibility (71, 73, 74, and 75) ; but the value of an unknown quantity may be reduced to $\frac{0}{0}$, by means of the presence of a common factor in the two terms of the fraction. This must be carefully examined (72) ;

(6.) That if all the second members of the proposed set of equations are zero, the values become also 0 ; that if to this hypothesis,

we add that in which the common denominator of the values of unknown quantities is 0, the number of the sets of values is infinite ; but that these values are subject to have a constant relation between them (78) ;

(7.) That, when the number of equations is greater than that of the unknown quantities, the problem is possible only, inasmuch as the values of the unknown quantities, determined by a number of equations equal to that of the unknown quantities, satisfy the other equations (79).

81. Enunciations of new problems susceptible of discussion, or whose resolution presents some interest.

Problem 15. *A banker has two kinds of money ; it requires a pieces of the first to make a crown ; and b pieces of the second to make the same sum. Some one demands c pieces for a crown. How many pieces of each kind must the banker give to satisfy him ?*

$$\left(\text{1st kind, } \frac{a(c-b)}{a-b} ; \text{2d kind } \frac{b(a-c)}{a-b} \right).$$

Problem 16. *To find the two adjacent sides of a rectangle, by supposing, (1.) that the two sides are to each other in a given ratio $m:n$; (2.) that if we alter the sides of this rectangle (by addition or subtraction) of the given quantities a and b, the surface will be altered by the quantity p.*

Supposing the sides altered by addition, we find

$$x = \frac{m(p-ab)}{na+mb}, \quad y = \frac{n(p-ab)}{na+mb}.$$

Problem 17. *It is required to find the property of three persons A, B, C, knowing, (1.) that the sum of the estates of A and of 1 times the property of B and C is equal to p ; (2.) that the sum of the estate of B and of m times the property of A and C, is equal to q ; (3.) that the sum of the estates of C and of n times the property of A and B, is equal to r.*

(This question may be very simply solved by introducing an unknown auxiliary quantity in the course of the calculation ; and this unknown quantity is the sum of the three estates.)

Problem 18. *To find the property of 6 persons, A, B, C, D, E, F, from the following conditions ; (1.) the sum of the estates of A and B is a ; that of the estates of C and D is b ; the sum of the*

estates E and F is c; (2.) the estate of A is worth m times the estate of C; the estate of D is worth n times the estate of E; the estate of F is worth p times the estate of B?

(This problem may be solved by means of a single equation with one unknown quantity.)

These problems are extracted from the Algebra of M. Lhuillier, of Geneva, a work distinguished for its selection of questions proposed as exercises.

Resolution of Problems and Equations of the Second Degree.

82. INTRODUCTION. When the enunciation of a problem leads to an equation, such as $a x^2 = b$, in which the unknown quantity is multiplied by itself, the equation is said to be of the *second degree*; and the principles established in the preceding articles are insufficient for its solution; but since, by the division of both its members by a , the equation becomes $x^2 = \frac{b}{a}$, it will be seen that the question is reduced to *finding a number, which, when multiplied by itself, shall produce the number expressed by $\frac{b}{a}$* ; this is what is proposed in the extraction of the square root.

Formation of the Square and Extraction of the Square Root of Algebraical Quantities.

83. Let us first take the case of a simple quantity, and in order to discover the process we must see how we have formed the square of a simple quantity. By the rules given (16) for the multiplication of simple quantities, we have

$$(5 a^2 b^3 c)^2 = 5 a^2 b^3 c \times 5 a^2 b^3 c = 25 a^4 b^6 c^2;$$

therefore, to raise a simple quantity to the square we must square its coefficient and double each of the exponents of its different letters. Then to return from the square to the square root of a simple quantity, we must, (1.) extract the square root of its coefficient; (2.) take the half of each of its exponents.

Thus, $\sqrt{64 a^6 b^4} = 8 a^3 b^2;$

for $(8 a^3 b^2)^2 = 8 a^3 b^2 \times 8 a^3 b^2 = 64 a^6 b^4.$

Bour. Alg.

In like manner,

$$\sqrt{625 a^3 b^8 c^6} = 25 a b^4 c^3,$$

for

$$(25 a b^4 c^3)^2 = 625 a^3 b^8 c^6.$$

From this rule it follows, that in order that a simple quantity may be the square of another simple quantity, *its coefficient must be a perfect square, and all its exponents must be even numbers.* Thus, $98 a b^4$ is not a perfect square, because 98 is not a perfect square and the exponent of a is not even.

In this case the quantity is placed under the sign $\sqrt{\quad}$, and is written thus, $\sqrt{98 a b^4}$. Expressions of this kind are called *radical or irrational quantities*, or, simply, *radicals of the second degree*.

84. These expressions may be often simplified by the use of the following principle. *The square root of the product of two or more factors is equal to the product of the square roots of these factors*, or, in algebraical language,

$$\sqrt{abcd \dots} = \sqrt{a} \cdot \sqrt{b} \cdot \sqrt{c} \cdot \sqrt{d} \dots$$

To demonstrate this principle, observe that, from the definition of a square root,

$$(\sqrt{abcd \dots})^2 = abcd \dots$$

On the other hand,

$$(\sqrt{a} \cdot \sqrt{b} \cdot \sqrt{c} \cdot \sqrt{d} \dots)^2 = (\sqrt{a})^2 \cdot (\sqrt{b})^2 \cdot (\sqrt{c})^2 \cdot (\sqrt{d})^2 \dots = abcd \dots$$

Then, since the squares of $\sqrt{abcd \dots}$ and $\sqrt{a} \cdot \sqrt{b} \cdot \sqrt{c} \cdot \sqrt{d} \dots$ are equal, these quantities themselves are equal.

This being premised, the above expression $\sqrt{98 a b^4}$ may be put under the form $\sqrt{49 b^4 \times 2 a}$, which is $\sqrt{49 b^4} \times \sqrt{2 a}$.

Now $\sqrt{49 b^4}$ (83) is reduced to $7 b^2$.

Then

$$\sqrt{98 a b^4} = 7 b^2 \cdot \sqrt{2 a}.$$

In like manner,

$$\sqrt{45 a^2 b^3 c^2 d} = \sqrt{9 a^2 b^2 c^2 \times 5 b d} = 3 a b c \cdot \sqrt{5 b d},$$

$$\sqrt{864 a^2 b^6 c^{11}} = \sqrt{144 a^2 b^4 c^{10} \times 6 b c} = 12 a b^2 c^5 \cdot \sqrt{6 b c}.$$

In general, to simplify an irrational simple quantity, *separate all its factors which are perfect squares, and extract their roots* (83); *place the product of all these roots before the radical sign, and all the factors under the radical sign, which are not perfect squares.*

In the expressions

$$7b^2\sqrt{2a}, 3abc\sqrt{5bd}, 12ab^3c^5\sqrt{6bc},$$

the quantities $7b^2$, $3abc$, $12ab^3c^5$, are called *the coefficients of the radical*.

85. We have hitherto paid no regard to the sign of the simple quantity. But since, in the solution of questions, we are led to the consideration of simple quantities preceded either by the sign $+$ or $-$, it is necessary to know how to proceed with quantities of this kind. Now, as the square of a simple quantity is the product of that simple quantity, multiplied by itself, it follows (62), that *whatever be its sign, its square is positive*; thus the square of $+5a^2b^3$ and the square of $-5a^2b^3$, are each $+25a^4b^6$.

Hence we may conclude, *that if a simple quantity be positive, its square root may be regarded either as positive or negative*; thus, $\sqrt{9a^4} = \pm 3a^2$; for $3a^2$ and $-3a^2$ raised to the second power, are each $9a^4$.

If the proposed simple quantity be negative, *the extraction of its square root is impossible*, because we have just seen that the square of every quantity, positive or negative, is positive. Thus, $\sqrt{-9}$, $\sqrt{-4a^2}$, $\sqrt{-5}$, are algebraical symbols representing operations which are impossible to be performed. They are known by the name of *imaginary expressions*, and they are symbols of absurdity, which are often met with in the resolution of problems of the second degree.

We may always, by extending the rules, give to these symbols the same simplifications as to irrational expressions upon which the necessary operations can be performed. Thus $\sqrt{-9}$ is reduced to

$$\sqrt{9} \cdot \sqrt{-1}, \text{ or } 3\sqrt{-1}.$$

$$\text{Also, } \sqrt{-4a^2} = \sqrt{4a^2} \cdot \sqrt{-1} = 2a\sqrt{-1}$$

$$\sqrt{-8a^2b} = \sqrt{4a^2 \times -2b} = 2a\sqrt{-2b} = 2a\sqrt{2b}\sqrt{-1}.$$

86. We now proceed to make known a *law of formation* for the square of any polynomial, from which we can deduce a rule for the extraction of the square root.

We have already seen (19) that the square of a binomial $a + b$, is equal to $a^2 + 2ab + b^2$. Let it be required to form the square of a trinomial $a + b + c$. And for the present designate $a + b$ by the single letter s , it becomes

$$(a + b + c)^2 = (s + c)^2 = s^2 + 2sc + c^2.$$

Now we have

$$s^2 = (a + b)^2 = a^2 + 2ab + b^2,$$

$$2sc = 2(a + b)c = 2ac + 2bc.$$

Therefore,

$$(a + b + c)^2 = a^2 + 2ab + b^2 + 2ac + 2bc + c^2.$$

That is, *the square of a trinomial is composed of the sum of the squares of the three terms, and of twice the product of the three terms multiplied together, two and two.*

This law of composition is true for any polynomial. For let us suppose it to be true for a polynomial containing a certain number of terms, it is true for a polynomial which has one term more.

For this purpose let $a + b + c + d + \dots + i + k$ be a polynomial of $m + 1$ terms, and designate the sum of the m first terms by s ; then $s + k$ is the given polynomial, and we have

$$(s + k)^2 = s^2 + 2sk + k^2;$$

or by substituting for s its value,

$$(s + k)^2 = (a + b + c + d + \dots + i)^2 + 2(a + b + c + d + \dots + i)k + k^2.$$

Now, the first part of this expression is composed, by hypothesis, *of the squares of all the terms of the first polynomial and twice the product of these terms multiplied two and two*; the second part contains *twice the product of each of the terms of the first multiplied by the new term k , and the third part is the square of that term.* Therefore the law of composition, enunciated above, is true for the polynomial of $m + 1$ terms, if it be true for that of m terms. But we have seen that it is true for the trinomial; therefore it is also true for the polynomial of four terms. Being true for that of four, it is also true for that of five terms, and so on. Therefore, it is general.

We may also enunciate this law in the following manner. *The square of a polynomial contains the square of its first term plus twice the product of its first term multiplied by the second, plus the square of the second; plus twice the product of each of the first two terms multiplied by the third, plus the square of the third; and also twice the product of each of the first three terms multiplied by the fourth, plus the square of the fourth; and so on. This enunciation, which is evidently comprehended in the first, will lead us more easily to the rule for the extraction of the square root of a polynomial.*

According to this law we shall find

$$\begin{aligned}(5a^3 - 4ab^2)^2 &= 25a^6 - 40a^4b^2 + 16a^2b^4, \\ (3a^3 - 2ab + 4b^2)^2 &= 9a^6 - 12a^4b + 4a^2b^2 + 24a^2b^3 - 16ab^3 + 16b^4, \\ &= 9a^6 - 12a^4b + 28a^2b^2 - 16ab^3 + 16b^4, \\ (5a^2b - 4abc + 6bc^2 - 3a^2c)^2 &= 25a^4b^2 - 40a^3b^2c + 76a^2b^2c^2 \\ &\quad - 48ab^2c^3 + 36b^2c^4 - 30a^4bc + 24a^3bc^2 - 36a^2bc^3 + 9a^4c^2.\end{aligned}$$

We will now proceed to the extraction of the square root.

87. Let us designate by N the polynomial whose root is to be found, and R the root, which we suppose determined; let these two polynomials be arranged with reference to the powers of one of the letters which they contain, as of a , for example.

The first two terms of N , thus arranged, will immediately give the first and second term of R ; for it evidently follows from the law of the formation of the square (86), (1.) *That the square of the first term of R contains the highest exponent of the letter a , which enters into the square of R .* (2.) *That twice the product of the first term of R by the second, contains also an exponent higher than that of the succeeding terms.* Thus the two parts which we have just mentioned, not being capable of reduction with any other terms, are necessarily the two terms of N which have the highest power of a , and the highest but one. Hence it follows, that if N be a perfect square. (1.) *Its first term must be a perfect square, and its root, extracted by the process of article 83, is the first term of R .* (2.) *Its second term must be divisible by twice the first term of R ; and by performing this division, the quotient will be the second term of R .*

To obtain the following terms, form the square of the binomial already found, and subtract it from N ; the remainder, which we call N' , contains still twice the product of the first term of R by the third, plus the product of the second by the third, together with a series of other parts. But double the product of the first term of R by the third must have a greater exponent of a than any of the following terms, and therefore cannot be reduced with any of these terms. Then this double product must be the first term of N' ; and the first term of this remainder must be divisible by twice the first term of R ; and if we perform this division, the quotient is the third term of R .

To obtain the next term, we must form twice the product of the first and second terms by the third, plus the square of the third, and

subtract their sum from the remainder N' ; this will give a new remainder N'' , which contains twice the product of the first term of R by the fourth, plus a series of other terms. But we may prove, as before, that *the first term of N'' is twice the product of the first term of R multiplied by the fourth; therefore the quotient of the first term of N'' divided by twice the first term of R is the fourth term of R ; and so on.*

N. B. It is indispensable, after having obtained the two first terms of the root, to subtract the square of this binomial from the polynomial N ; for, commonly, the square of the second term of R contains the same power of a as twice the product of the first term by the third, and therefore these two terms may be reduced to one. It is not, therefore, till after this square has been subtracted, that we may be sure that the first term of the remainder is equal to twice the product of the first term of R by the third. The same remark is applicable to the other terms.

It is left for the learner to deduce from the preceding reasonings the general process for the extraction of the square root of a polynomial. It is only necessary to bring together all the portions in *italics*.

We now proceed to apply this rule to a particular example. Let it be required to extract the square root of the polynomial,

$$\begin{array}{r}
 49a^2b^2 - 24ab^3 + 25a^4 - 30a^3b + 16b^4 \\
 \left. \begin{array}{l} 25a^4 - 30a^3b + 49a^2b^2 - 24ab^3 + 16b^4 \\ - 25a^4 + 30a^3b - 9a^2b^2 \end{array} \right\} \begin{array}{l} 5a^2 - 3ab + 4b^2 \\ \hline 10a^2 \end{array} \\
 \hline
 \text{1st rem.} \quad 40a^2b^2 - 24ab^3 + 16b^4 \\
 \quad \quad \quad - 40a^2b^2 + 24ab^3 - 16b^4 \\
 \hline
 \text{2d rem.} \quad \quad \quad \quad \quad \quad \quad 0
 \end{array}$$

After having arranged the expression with reference to the powers of a , we extract the square root of $25a^4$, which gives $5a^2$, and write this on the right hand of the given polynomial; then divide the second term $-30a^3b^2$ by $10a^2$, or twice $5a^2$ (which is written underneath $5a^2$); the quotient is $-3ab$, which is placed at the right of $5a^2$. The two first terms of the root are, therefore, $5a^2 - 3ab$. The square of this binomial is

$$25a^4 - 30a^3b + 9a^2b^2,$$

which, subtracted from the given polynomial, gives a remainder whose first terms is $40a^2b^2$. Dividing this first term by $10a^2$, or

twice $5a^2$, we obtain the quotient $+4b^2$, this is the third term of the root, and is written on the right of the two first terms. Form twice the product of

$$5a^2 - 3ab \text{ by } 4b^2,$$

and the square of $4b^2$. Their sum is

$$40a^2b^2 - 24ab^3 + 16b^4,$$

which, when subtracted from the first remainder, leaves nothing. Therefore, $5a^2 - 3ab + 4b^2$ is the root sought.

Beginners may apply this rule to the squares developed in article 86.

88. If the given polynomial contain several terms which have the same power of the principal letter, it must be arranged according to the directions given in division (29), and the above process must be applied by considering, as one part, the *algebraic sum* of the terms which have the same power of the principal letter, and substituting in the enunciation of the process, instead of the words *first term* of the polynomial, *first term* of the remainder, *first term* of the root, the expression *first portion* of the polynomial or *portion affected with the highest power*, *first portion* of the remainder, *first portion* of the root. Examples of this nature very rarely present themselves.

89. We will terminate this subject by the following remarks.

(1.) A binomial can never be a perfect square, because the square of the most simple polynomial, that is, of a binomial, contains three distinct parts, which do not admit of any reduction. Thus the expression $a^2 + b^2$ is not a perfect square; it wants the term $\pm 2ab$ to make it the square of $a \pm b$.

(2.) In order that a trinomial may be a perfect square, it must, when arranged, have its extreme terms squares, and the middle term equal to twice the product of the square roots of the two others. The square root of such a trinomial may be obtained immediately; *extract the square roots of the extreme terms, and give to these roots the same or contrary signs, according as the middle term is positive or negative. See then if twice the product of these roots will give the middle term of the trinomial.* Thus

$$9a^6 - 48a^4b^2 + 64a^2b^4$$

has for its square root

$$\sqrt{9a^6} - \sqrt{64a^2b^4}, \text{ that is, } 3a^3 - 8ab^2,$$

since $3a^2 \times -16ab^2 = -48a^4b^2$.

$4a^2 + 12ab - 9b^2$ is not a perfect square; for although $4a^2$ and $9b^2$ are the squares of $2a$ and $3b$, and $12ab = 2a \times 6b$, yet $-9b^2$ is not a perfect square.

(3.) Whenever, in the series of operations which the general process contains, the first term of one of the remainders is not exactly divisible by twice the first term of the root, we may conclude that the given polynomial is not a perfect square. This is an evident consequence of the reasoning by which we arrived at the process.

(4.) The simplifications of article 84 may often be applied to the square roots of polynomials which are not perfect squares.

For example, let there be the expression

$$\sqrt{a^3b + 4a^2b^2 + 4ab^3};$$

the quantity under the radical sign is not a perfect square, but it may be put under the form $ab(a^2 + 4ab + 4b^2)$. Now the factor within the parenthesis is evidently the square of $a + 2b$; whence we conclude (84)

$$\sqrt{a^3b + 4a^2b^2 + 4ab^3} = (a + 2b)\sqrt{ab}.$$

Calculus of Radicals of the Second Degree.

90. Since the extraction of the square root gives rise to new algebraical expressions, such as $\sqrt{a}$, $3\sqrt{b}$, $7\sqrt{2}$, which are called *irrational quantities*, or *radical quantities of the second degree*, it is necessary to lay down the rules for performing on these expressions the four fundamental operations.

Two radicals of the second degree are said to be *similar*, when the quantity under the radical sign is the same in both. Thus, $3a\sqrt{b}$ and $5c\sqrt{b}$, are similar radicals; so also are $9\sqrt{2}$ and $7\sqrt{2}$.

Addition and Subtraction. To add or subtract two similar radicals, add or subtract the two coefficients, and apply the common radical sign to their sum or difference. Thus,

$$3a\sqrt{b} + 5c\sqrt{b} = (3a + 5c)\sqrt{b};$$

$$3a\sqrt{b} - 5c\sqrt{b} = (3a - 5c)\sqrt{b}.$$

In like manner,

$$7\sqrt{2a} + 3\sqrt{2a} = 10\sqrt{2a}; \quad 7\sqrt{2a} - 3\sqrt{2a} = 4\sqrt{2a},$$

Two radicals, which do not at first sight appear similar, may become so by the simplifications of article 84.

For example,

$$\sqrt{48a^3b} + b\sqrt{75a} = 4b\sqrt{3a} + 5b\sqrt{3a} = 9b\sqrt{3a},$$

$$2\sqrt{45} - 3\sqrt{5} = 6\sqrt{5} - 3\sqrt{5} = 3\sqrt{5}.$$

If the radicals are not similar, the addition or subtraction can only be indicated. Thus in order to add $3\sqrt{b}$ to $5\sqrt{a}$, we write simply $5\sqrt{a} + 3\sqrt{b}$.

Multiplication. To multiply two radicals by each other, multiply together the two quantities under the radical sign, and place the radical sign over their product. Thus,

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}.$$

This is the converse of the principle of article 84.

If there are coefficients, multiply them together, and write their product before the radical sign.

For example,

$$3\sqrt{5ab} \times 4\sqrt{20a} = 12\sqrt{100a^2b} = 120a\sqrt{b},$$

$$2a\sqrt{bc} \times 3a\sqrt{bc} = 6a^2\sqrt{b^2c^2} = 6a^2bc,$$

$$2a\sqrt{a^2+b^2} \times -3a\sqrt{a^2+b^2} = -6a^2(a^2+b^2).$$

Division. To divide one radical by another, divide the two quantities which are under the radical sign by one another, and place the quotient under the radical sign. Thus,

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}.$$

For the square of each of these expressions is $\frac{a}{b}$, therefore they are equal. If there are coefficients, we write their quotient as the coefficient of the radical.

For example,

$$\frac{5a\sqrt{b}}{2b\sqrt{c}} = \frac{5a}{2b} \sqrt{\frac{b}{c}},$$

$$\frac{12ac\sqrt{6bc}}{4c\sqrt{2b}} = 3a \sqrt{\frac{6bc}{2b}} = 3a\sqrt{3c}.$$

91. There are two transformations of frequent use in finding the numerical values of radicals.

The first consists in placing the coefficient under the radical sign. For example, let there be the expression $3a\sqrt{5b}$;

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we see that this is reduced to

$$\sqrt{9a^2} \times \sqrt{5b}, \text{ or } \sqrt{9a^2 \cdot 5b} = \sqrt{45a^2b},$$

by applying the rule for multiplication to the two radicals. Thus the square of the coefficient may be placed under the radical sign.

The principal use of this transformation is the following. Suppose it were required to find, *within a unit*, the value of $6\sqrt{13}$. Since 13 is not a perfect square, only an approximate value of this expression can be obtained. This root is equal to 3 plus a certain fraction; but by multiplying this by 6, we have 18 plus the product of this fraction by 6; so that the total result may give a whole number greater than 18. The only method of determining this whole part exactly, is by putting $6\sqrt{13}$ under the form

$$\sqrt{6^2 \times 13} = \sqrt{36 \times 13} = \sqrt{468}.$$

The square root of 468 is 21 plus a fraction; therefore, $6\sqrt{13}$ is 21 plus a fraction.

In the same manner, $12\sqrt{7} = 31$ plus a fraction.

The object of the second transformation is to make the denominators of such expressions as

$$\frac{a}{p + \sqrt{q}}, \quad \frac{a}{p - \sqrt{q}},$$

rational, a and p being any entire numbers whatever, and q any number which is not a perfect square. Expressions of this kind often occur in the resolution of problems of the second degree. This purpose is effected by multiplying both terms of the fraction by $p - \sqrt{q}$, if the denominator is $p + \sqrt{q}$, and by $p + \sqrt{q}$, if the denominator is $p - \sqrt{q}$. For, since the sum of two quantities multiplied by their difference is equal to the difference of their squares, we have, by this multiplication,

$$\begin{aligned} \frac{a}{p + \sqrt{q}} &= \frac{a(p - \sqrt{q})}{(p + \sqrt{q})(p - \sqrt{q})} = \frac{ap - a\sqrt{q}}{p^2 - q}, \\ \frac{a}{p - \sqrt{q}} &= \frac{a(p + \sqrt{q})}{(p - \sqrt{q})(p + \sqrt{q})} = \frac{ap + a\sqrt{q}}{p^2 - q}, \end{aligned}$$

expressions which have rational denominators.

To show the utility of this transformation, suppose it required to find the approximate value of the expression

$$\frac{7}{3 - \sqrt{5}}.$$

This is reduced to

$$\frac{7(3 + \sqrt{5})}{9 - 5}, \text{ or } \frac{21 + 7\sqrt{5}}{4}.$$

Now $7\sqrt{5}$ is the same as

$$\sqrt{7^2 \times 5} \text{ or } \sqrt{245},$$

the value of which is 15 to within 1. Therefore,

$$\frac{7}{3 - \sqrt{5}} = \frac{21 + 15 + \text{a fraction}}{4} = \frac{36}{4} = 9$$

to within a fraction indicated by $\frac{1}{4}$, so that 9 is the value within $\frac{1}{4}$.

If a more exact value of this expression is required, it is sufficient to calculate $\sqrt{245}$ to the required degree of approximation, to add 21 to the root obtained, and then divide the sum by 4.

Required the value of the expression

$$\frac{7\sqrt{5}}{\sqrt{11} + \sqrt{3}}$$

to within 0.01. We have,

$$\frac{7\sqrt{5}}{\sqrt{11} + \sqrt{3}} = \frac{7\sqrt{5}(\sqrt{11} - \sqrt{3})}{11 - 3} = \frac{7\sqrt{55} - 7\sqrt{15}}{8};$$

now

$$7\sqrt{55} = \sqrt{55 \times 49} = \sqrt{2695} = 51.91 \text{ to within } 0.01.$$

$$7\sqrt{15} = \sqrt{15 \times 49} = \sqrt{735} = 27.11 \dots;$$

so that

$$\frac{7\sqrt{5}}{\sqrt{11} + \sqrt{3}} = \frac{51.91 - 27.11}{8} = \frac{24.80}{8} = 3.10;$$

the result required is 3.10, which is exact to within $\frac{1}{10}$, as may easily be seen.

By a process analogous to the preceding, we shall find

$$\frac{3 + 2\sqrt{7}}{5\sqrt{12} - 6\sqrt{6}} = 3.159 \text{ to within } 0.001.$$

N. B. We might have calculated the value of this kind of expressions by finding an approximate value of each of the radicals which enter into the numerator as well as into the denominator. But as this would not give an exact value of the denominator, no very precise idea could be formed of the degree of approximation which would be obtained; while, by the method just described, the denominator is made rational, and we know upon what to depend for the required degree of approximation.

Having established the principles on which the extraction of the square roots of numbers and algebraical quantities depend, we can now proceed to the solution of problems of the second degree.

Problems and Equations of the Second Degree.

92. We distinguish two kinds of equations of the second degree; equations of *two terms*, or *incomplete*, and equations of *three terms*, or *complete*.

The first are those which contain only terms affected with the square of the unknown quantity and terms altogether known. Such are the equations

$$\begin{aligned} 3x^2 &= 5, \\ 4x^2 - 7 &= 3x^2 + 9, \\ \frac{1}{2}x^2 - 3 + \frac{1}{12}x^2 &= \frac{7}{24} - x^2 + \frac{29}{24}; \end{aligned}$$

they are called equations of *two terms*, because, by means of the general transformations of articles 43 and 44, they may always be reduced to the form $ax^2 = b$. For let us consider the last of these equations and the most complicated of the three; we have, in the first place, by freeing it from denominators,

$$8x^2 - 72 + 10x^2 = 7 - 24x^2 + 299,$$

or by transposing and reducing,

$$42x^2 = 378.$$

Equations of three terms, or complete equations, are those which, besides the square of the unknown quantity, contain the first power of this quantity. Such are the equations

$$\begin{aligned} 5x^2 - 7x &= 34, \\ \frac{1}{2}x^2 - \frac{1}{2}x + \frac{3}{4} &= 8 - \frac{2}{3}x - x^2 + \frac{17}{12}; \end{aligned}$$

which can always be reduced to the form $ax^2 + bx = c$ by means of the transformations already mentioned.

Equation of Two Terms. The solution of the equation $ax^2 = b$ presents no difficulty. From it we deduce, in the first place,

$$x^2 = \frac{b}{a}, \text{ whence } x = \sqrt{\frac{b}{a}}.$$

If $\frac{b}{a}$ be some number, whole or fractional, its square root may be obtained, either exactly or by approximation. If $\frac{b}{a}$ be an algebraic quantity, we apply to it the principles established for algebraic quantities.

Nevertheless we observe, that as the squares of $+m$ and $-m$ are equally $+m^2$, so, in like manner,

$$\left(+\sqrt{\frac{b}{a}}\right)^2$$

and

$$\left(-\sqrt{\frac{b}{a}}\right)^2$$

give equally for a result $\frac{b}{a}$. So that the equation is in reality susceptible of two solutions,

$$x = +\sqrt{\frac{b}{a}} \text{ and } x = -\sqrt{\frac{b}{a}}.$$

For if we substitute each of these values in the equation $ax^2 = b$, it becomes

$$a \times \left(+\sqrt{\frac{b}{a}}\right)^2 = b, \text{ or } a \times \frac{b}{a} = b, \text{ or } b = b,$$

and

$$a \times \left(-\sqrt{\frac{b}{a}}\right)^2 = b, \text{ or } a \times \frac{b}{a} = b, \text{ or } b = b.$$

For the first example let there be the equation

$$4x^2 - 7 = 3x^2 + 9.$$

We find, by transposition,

$$x^2 = 16, \text{ whence } x = \pm \sqrt{16} = \pm 4.$$

For a second example let us take the equation

$$\frac{1}{3}x^2 - 3 + \frac{1}{12}x^2 = \frac{7}{12} - x^2 + \frac{22}{3}.$$

We have already seen (92) that this equation may be reduced to $42x^2 = 378$, whence $x^2 = \frac{378}{42} = 9$, and $x = \pm 3$.

Again, let there be the equation $3x^2 = 5$, we deduce from it

$$x = \pm \sqrt{\frac{5}{3}} = \pm \frac{1}{3}\sqrt{15}.$$

As 15 is not a perfect square, we can only determine the values of x by approximation.

93. *Complete Equation of the Second Degree.* To resolve the equation $ax^2 + bx = c$, we begin by dividing the two members by the coefficient of x^2 ; it becomes

$$x^2 + \frac{b}{a}x = \frac{c}{a}, \text{ or } x^2 + px = q,$$

by putting, for the sake of greater simplicity,

$$\frac{b}{a} = p \text{ and } \frac{c}{a} = q.$$

This being premised, we remark, that if we can alter the first member so as to make it the square of a binomial, a simple extraction of the square root will reduce the equation to an equation of the first degree. Now by comparing the first member with the square of the binomial $x + a$, or with $x^2 + 2ax + a^2$, we see that $x^2 + px$ is composed of the square of the first term; it follows that x plus twice the product of this first term x by a second term which is then $\frac{p}{2}$ (for we have $px = 2 \times \frac{p}{2}x$); whence, if to $x^2 + px$ we add the square of $\frac{p}{2}$ or $\frac{p^2}{4}$, the first member of the equation becomes the square of $x + \frac{p}{2}$. But in order that we may not destroy the equality of the two members, we must likewise add $\frac{p^2}{4}$ to the second member.

By this transformation it becomes

$$x^2 + px + \frac{p^2}{4} = \frac{p^2}{4} + q;$$

whence, by extracting the square root,

$$x + \frac{p}{2} = \pm \sqrt{\frac{p^2}{4} + q}.$$

(The double sign $\pm$ is put here, because the square of

$$+ \sqrt{\frac{p^2}{4} + q} \text{ and } - \sqrt{\frac{p^2}{4} + q} \text{ are each } \frac{p^2}{4} + q.)$$

Deducing the value of x from the last equation, we obtain

$$x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}.$$

Whence results this general rule for the solution of a complete equation of the second degree. After having *reduced the equation to the form* $x^2 + px = q$; *add to the two members the square of half the coefficient of* x *in the second term, extract the square root of both members, and place the double sign* $\pm$ *before the square root of the second member, and, lastly, deduce the values of* x *from the equation thus obtained.*

The double value of x , which we have obtained by this method, may be thus expressed in ordinary language; *half the coefficient of* x *taken with the contrary sign plus or minus the square root of the known term, increased by the square of half the coefficient of* x .

94. As a first example, let there be the equation

$$\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{2} = 8 - \frac{1}{2}x - x^2 + \frac{1}{11}.$$

By freeing it from denominators it becomes

$$10x^2 - 6x + 9 = 96 - 8x - 12x^2 + 273,$$

or by transposing and reducing

$$22x^2 + 2x = 360;$$

dividing both members by 22,

$$x^2 + \frac{1}{11}x = \frac{360}{22}.$$

Now by adding to each member $(\frac{1}{22})^2$, the equation becomes

$$x^2 + \frac{1}{11}x + (\frac{1}{22})^2 = \frac{360}{22} + (\frac{1}{22})^2.$$

Whence, by extracting the square root,

$$x + \frac{1}{22} = \pm \sqrt{\frac{360}{22} + (\frac{1}{22})^2}.$$

Then

$$x = -\frac{1}{22} \pm \sqrt{\frac{360}{22} + (\frac{1}{22})^2};$$

a result which is conformable to the expression, given above for the double value of x .

It remains to perform the numerical calculations. First, we must reduce $\frac{360}{22} + (\frac{1}{22})^2$ to a common denominator.

Now,

$$\frac{360}{22} + (\frac{1}{22})^2 = \frac{360 \times 22 + 1}{(22)^2} = \frac{7921}{(22)^2}.$$

By extracting the square root of 7921 we find exactly 89; so that

$$\sqrt{\frac{360}{22} + (\frac{1}{22})^2} = \frac{89}{22}.$$

Therefore

$$x = -\frac{1}{22} \pm \frac{89}{22}.$$

By separating the two values it becomes

$$x = -\frac{1}{22} + \frac{89}{22} = \frac{88}{22} = 4,$$

$$x = -\frac{1}{22} - \frac{89}{22} = -\frac{90}{22} = -\frac{45}{11}.$$

So that of the two values of x which will satisfy the proposed equation one is a positive whole number, the other a negative fractional one.

For a second example, let there be the equation

$$6x^2 - 37x = -57,$$

which is reduced to

$$x^2 - \frac{37}{6}x = -\frac{57}{6}.$$

If we add to both members $(\frac{37}{12})^2$, it becomes

$$x^2 - \frac{37}{6}x + (\frac{37}{12})^2 = -\frac{57}{6} + (\frac{37}{12})^2;$$

whence, by extracting the square root,

$$x - \frac{37}{12} = \pm \sqrt{-\frac{57}{6} + (\frac{37}{12})^2};$$

and lastly,

$$x = \frac{37}{12} \pm \sqrt{-\frac{57}{6} + (\frac{37}{12})^2}.$$

In order to reduce $(\frac{37}{12})^2 - \frac{57}{6}$ to a whole number, we see that $(12)^2 = 12 \times 12 = 6 \times 24$; so that it is sufficient to multiply 57 by 24, then 37 by itself, and then divide the difference between the two products by $(12)^2$.

Now $37 \times 37 = 1369$, $57 \times 24 = 1368$,
then

$$(\frac{37}{12})^2 - \frac{57}{6} = \frac{1}{12^2};$$

an expression, the square root of which is $\frac{1}{12}$.

Therefore $x = \frac{37}{12} \pm \frac{1}{12}$,

$$\text{or } \begin{cases} x = \frac{37}{12} + \frac{1}{12} = \frac{38}{12} = \frac{19}{6}, \\ x = \frac{37}{12} - \frac{1}{12} = \frac{36}{12} = 3. \end{cases}$$

This example is remarkable, inasmuch as each of the values of x are positive, and are direct answers to the question, of which the proposed equation is the algebraical translation.

Let there be the literal equation

$$4a^2 - 2x^2 + 2ax = 18ab - 18b^2.$$

By transposing and changing its signs, and dividing by 2, it becomes

$$x^2 - ax = 2a^2 - 9ab + 9b^2;$$

by completing the square,

$$x^2 - ax + \frac{a^2}{4} = \frac{9a^2}{4} - 9ab + 9b^2;$$

extracting the root,

$$x - \frac{a}{2} = \pm \sqrt{\frac{9a^2}{4} - 9ab + 9b^2}.$$

Now

$$\frac{9a^2}{4} - 9ab + 9b^2$$

has evidently for its root

$$(\frac{3a}{2} - 3b);$$

then
$$x = \frac{a}{2} \pm \left(\frac{3a}{2} - 3b \right)$$

whence
$$\begin{cases} x = 2a - 3b, \\ x = -a + 3b. \end{cases}$$

These values will both be positive if

$$2a > 3b \text{ and } 3b > a,$$

that is, if
$$b > \frac{a}{3} \text{ and } < \frac{2a}{3}.$$

The following equations are proposed as an exercise for the student ;

$$x^2 - 7x + 10 = 0 ;$$

values
$$x = 2, \quad x = 5 ;$$

$$\frac{1}{3}x - 4 - x^2 + 2x - \frac{4}{5}x^2 = 45 - 3x^2 + 4x,$$

values
$$\begin{cases} x = 7,12 \\ x = -5,75 \end{cases} \text{ to within } 00,1 ;$$

$$a^2 + b^2 - 2bx + x^2 = \frac{m^2 x^2}{n^2},$$

gives
$$x = \frac{n}{n^2 - m^2} (bn \pm \sqrt{a^2 m^2 + b^2 m^2 - a^2 n^2}).$$

95. The equation $ax^2 + bx = c$ may be resolved without dividing by the coefficient of x^2 , but the transformations are more complicated.

The term ax^2 may be put under the form $(x\sqrt{a})^2$, and the term bx under the form

$$2x\sqrt{a} \times \frac{b}{2\sqrt{a}} ;$$

whence $ax^2 + bx$ are the two first terms of the square of

$$x\sqrt{a} + \frac{b}{2\sqrt{a}} ;$$

So that by adding to both members $\left(\frac{b}{2\sqrt{a}}\right)^2$ or $\frac{b^2}{4a}$, the first member becomes a perfect square.

Performing this transformation the equation becomes

$$ax^2 + bx + \frac{b^2}{4a} = c + \frac{b^2}{4a} ;$$

extracting the root,

$$x\sqrt{a} + \frac{b}{2\sqrt{a}} = \pm \sqrt{c + \frac{b^2}{4a}},$$

transposing,

$$x\sqrt{a} = -\frac{b}{2\sqrt{a}} \pm \sqrt{c + \frac{b^2}{4a}}.$$

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Dividing both members by $\sqrt{a}$, and observing

(1) that
$$-\frac{b}{2\sqrt{a}}$$

divided by $\sqrt{a}$

$$= -\frac{b}{2(\sqrt{a})^3} = -\frac{b}{2a};$$

(2) that

$$\sqrt{c + \frac{b^2}{4a}}$$

divided by $\sqrt{a}$

$$= \frac{1}{\sqrt{a}} \sqrt{c + \frac{b^2}{4a}} = \sqrt{\frac{c}{a} + \frac{b^2}{4a^2}} \quad (90.)$$

We obtain

$$x = -\frac{b}{2a} \pm \sqrt{\frac{c}{a} + \frac{b^2}{4a^2}},$$

or

$$x = \frac{-b \pm \sqrt{4ac + b^2}}{2a},$$

the same result which was obtained more easily by putting the equation under the form

$$x^2 + \frac{b}{a}x = \frac{c}{a}.$$

96. We proceed to apply the preceding principles to the solution of some problems.

Problem 1. *To find a number such, that twice its square added to three times the number, may be 65.*

Let x be the unknown number. The equation of the problem is

$$2x^2 + 3x = 65.$$

Whence

$$x = -\frac{3}{4} \pm \sqrt{\frac{65}{2} + \frac{9}{16}} = -\frac{3}{4} \pm \frac{23}{4}.$$

Therefore

$$x = -\frac{3}{4} + \frac{23}{4} = 5,$$

and

$$x = -\frac{3}{4} - \frac{23}{4} = -\frac{26}{4}.$$

The first value is the one which answers the question in the sense in which it was proposed. For

$$2 \times (5)^2 + 3 \times 5 = 50 + 15 = 65.$$

To interpret the second, observe, that if we change x into $-x$, in the equation

$$2x^2 + 3x = 65,$$

the term $3x$ only changes its sign, since $(-x)^2 = x^2$. Instead of

$$x = -\frac{3}{2} \pm \frac{3}{2}$$

we have

$$x = \frac{3}{2} \pm \frac{3}{2};$$

that is,

$$x = \frac{3}{2}, \text{ or } x = -5,$$

which values differ from the former only in their signs. We may therefore say that the negative solution $-\frac{3}{2}$, considered independently of its sign, is the answer to the following question. *To find a number such that twice its square diminished by three times the number is 65?* For we have

$$2 \times \left(\frac{13}{2}\right)^2 - 3 \times \frac{13}{2} = \frac{169 - 39}{2} = 65.$$

Problem 2. *A man bought a certain number of ells of cloth for 240 francs. If, with the same sum, he had bought 3 ells less, it would have cost him 4 francs an ell more. How many ells did he buy?*

Let x be the number of ells; then $\frac{240}{x}$ is the price of each. If for 240 francs he had bought $x - 3$ ells, each ell would have cost $\frac{240}{x-3}$. But according to the enunciation the first price is greater than the second by 4; we have then the equation

$$\frac{240}{x-3} - \frac{240}{x} = 4;$$

whence we deduce

$$x^2 - 3x = 180,$$

$$x = \frac{3}{2} \pm \sqrt{\frac{9}{4} + 180} = \frac{3 \pm 27}{2};$$

then

$$x = 15, \text{ and } x = -12.$$

The first value $x = 15$ satisfies the enunciation; since 15 ells for 240 francs gives $\frac{240}{15}$ or 16 francs, for the price of 1 ell; and 12 ells for 240 francs gives for the price of an ell $\frac{240}{12}$ or 20 francs, the second of which is greater than the first by 4.

As to the second solution, we can form a new enunciation to which it will correspond. For by going back to the original equation, and changing x into $-x$, it becomes

$$\frac{240}{-x-3} - \frac{240}{-x} = 4, \text{ or } \frac{240}{x} - \frac{240}{x+3} = 4,$$

which may be considered as the algebraical translation of the following problem.

A person buys a certain number of ells of cloth for 240 francs. If he had paid the same sum for 3 ells more, each ell would have cost 4 francs less. Required the number of ells which he bought.

By resolving the equation of this new problem we shall evidently find $x = 12$, and $x = -15$, for the equation becomes

$$x^2 + 3x = 180,$$

instead of

$$x - 3x = 180.$$

Problem 3. *A merchant discounts two bills, one of 8776 francs, payable in nine months, the other of 7488 francs, payable in eight months; he pays for the first 1200 francs more than for the second. What is the rate of interest at which he discounts it?*

To make the calculation more simple, let x be the interest of 100 francs for a month, and $12x$ that for a year; $9x$ and $8x$ are the interest of it for nine and eight months; therefore $100 + 9x$, and $100 + 8x$, are what 100 francs amount to at the end of nine and eight months. So that in order to determine the actual value of the two notes of 8776 francs, 7488 francs, we must make the proportions

$$100 + 9x : 100 :: 8776 : \frac{877600}{100 + 9x},$$

$$100 + 8x : 100 :: 7488 : \frac{748800}{100 + 8x};$$

and the fourth terms express what the merchant has paid for each note.

By the enunciation, we have the equation

$$\frac{877600}{100 + 9x} - \frac{748800}{100 + 8x} = 1200;$$

which, being freed from denominators and reduced, becomes

$$216x^2 + 4396x = 2200,$$

whence

$$x = -\frac{2198}{216} \pm \sqrt{\frac{2200}{216} + \left(\frac{2198}{216}\right)^2}.$$

Reducing the two terms under the radical to the same denominator,

$$x = \frac{-2198 \pm \sqrt{5306404}}{216},$$

or multiplying by 12

$$12x = \frac{-2198 \pm \sqrt{5306404}}{18}.$$

To obtain the value of $12x$ to within 0,01 it is sufficient to extract the square root of 5306404 to within 0,1, since it must afterwards be divided by 18. This root is 2303,5; then

$$12x = \frac{-2198 \pm 2303,5}{18} = \frac{105,5}{18} = 5,86,$$

and

$$12x = \frac{-4501,5}{18} = -250,08.$$

The positive value 5,86 is the rate of interest required.

As to the negative solution it can only be considered as united with the first by the same equation of the second degree. For if we go back to the enunciation, and change x into $-x$, we shall find it difficult to translate the new equation into an enunciation analogous to that of the proposed problem.

Problem 4. A man buys a horse, which he sells some time after for 24 guineas. By this bargain he loses as many guineas in the hundred as his horse cost him at first. Required the price of the horse?

Let x be the number of guineas which the horse cost, $x - 24$ is a first expression for the loss.

But since he loses as many guineas in the hundred as there are units in x , on every guinea he loses $\frac{x}{100}$, and on x guineas he loses $\frac{x^2}{100}$. We have then the equation,

$$\frac{x^2}{100} = x - 24,$$

whence

$$x^2 - 100x = -2400,$$

$$x = 50 \pm \sqrt{2500 - 2400} = 50 \pm 10.$$

Then

$$x = 60, \text{ and } x = 40.$$

Both values equally satisfy the question. For suppose the price of the horse to be 60; since 24 is what he sells it for, his loss is 36. But he loses 60 per cent. on the sum which the horse cost, that is, $\frac{60}{100} \times 60$, which is also 36.

Again, let 40 be the price. Then the loss is 16, and moreover he loses 40 per cent, on 40 which is $40 \times \frac{40}{100}$, which equals 16.

General Discussion of the Equation of the Second Degree.

Hitherto we have only solved problems of the second degree, in which the given quantities were expressed by particular numbers. But in order to solve general problems, and interpret all the results which can be obtained, by assigning particular values to the given quantities, we must resume the most general equation of the second degree, and examine the circumstances which arise out of every hypothesis which can be made as to the value of the coefficients. This is the object of *the discussion of equations of the second degree*.

97. Before entering on the discussion, we shall point out another method of solving equations of the second degree, which will lead to some important properties which belong to the values of the unknown quantity.

We have already seen (93) that every equation of the second degree can be reduced to the form

$$x^2 + p x = q \dots \dots (1),$$

p and q being either numerical or algebraical quantities, whole or fractional, positive or negative.

This being premised, if, in order to make the first member a perfect square, we add $\frac{p^2}{4}$ to both members, the equation becomes

$$x^2 + p x + \frac{p^2}{4} = q + \frac{p^2}{4},$$

or
$$\left(x + \frac{p}{2}\right)^2 = q + \frac{p^2}{4}.$$

Whatever may be the value of $q + \frac{p^2}{4}$, we may always designate its square root by m ; then the equation becomes

$$\left(x + \frac{p}{2}\right)^2 = m^2, \text{ or } \left(x + \frac{p}{2}\right)^2 - m^2 = 0.$$

The first member of this equation being the difference of two squares, may (19) be put under the form

$$\left(x + \frac{p}{2} - m\right) \left(x + \frac{p}{2} + m\right) = 0 \dots \dots (2),$$

the first member of which is the product of two factors, and the second is 0. Now the product may be made equal to 0, and the equation (2) satisfied, either by making

$$x + \frac{p}{2} - m = 0, \text{ whence } x = -\frac{p}{2} + m,$$

or by making $x + \frac{p}{2} + m = 0$,

whence $x = -\frac{p}{2} - m$;

or substituting for m its value

$$x = -\frac{p}{2} + \sqrt{q + \frac{p^2}{4}},$$

$$x = -\frac{p}{2} - \sqrt{q + \frac{p^2}{4}}.$$

It is moreover evident that the first member of equation (2) cannot be made equal to 0, except by giving x a value, which destroys one of its factors.

Then, since equation (2) is derived from (1), and the reverse, every equation of the second degree admits of two values of the unknown quantity, and cannot have more.

This method of solution, which, perhaps, is a little longer than the first, has the advantage of showing more clearly that there are two values of the unknown quantity and only two.

98. These values have some very remarkable properties.

(1) since the equation

$$x^2 + p x = q, \text{ or } x^2 + p x - q = 0,$$

is reduced, by a series of transformations, to the form

$$\left(x + \frac{p}{2} - m\right) \left(x + \frac{p}{2} + m\right) = 0;$$

where

$$m = \sqrt{q + \frac{p^2}{4}}$$

it follows that the first member

$$x^2 + p x - q$$

of every equation of the second degree, whose second member is 0, is composed of two binomial factors of the first degree in terms of x , x being a term common to both, and their second terms being the two values of x , taken with contrary signs.

From this property, by which the equation itself may be found, when the values of the unknown quantity are given, these values have been called the *roots* of the equation.

Let there be the equation

$$x^2 + 3 x - 28 = 0,$$

which being resolved gives

$$x = 4, \text{ and } x = -7,$$

the first member results from the product of

$$(x - 4)(x + 7);$$

for this product is

$$x^2 - 4x + 7x - 28 = x^2 + 3x - 28.$$

(2.) If we designate the two roots by x' and x'' , according to the preceding property,

$$x^2 + px - q = (x - x')(x - x''),$$

or, by performing the operations,

$$x^2 + px + q = x^2 - (x' + x'')x + x'x''.$$

By comparing the analogous terms in the two members we find

$$x' + x'' = -p, \quad x'x'' = -q.$$

Therefore, (1) *the algebraic sum of the two roots is equal to the coefficient of the second term, taken with the contrary sign.* (2) *the product of the two roots is equal to the last term of the equation or the known term, when transposed to the first member of the equation.*

These two properties may be verified from the general expression of the two roots.

If we add

$$x' = -\frac{p}{2} + \sqrt{q + \frac{p^2}{4}}, \quad \text{to} \quad x'' = -\frac{p}{2} - \sqrt{q + \frac{p^2}{4}},$$

we obtain

$$x' + x'' = -\frac{p}{2} + \sqrt{q + \frac{p^2}{4}} - \frac{p}{2} - \sqrt{q + \frac{p^2}{4}} = -p.$$

Now by multiplying the same equations, we obtain

$$\begin{aligned} x'x'' &= \left(-\frac{p}{2} + \sqrt{q + \frac{p^2}{4}}\right) \left(-\frac{p}{2} - \sqrt{q + \frac{p^2}{4}}\right) \\ &= \frac{p^2}{4} - \left(q + \frac{p^2}{4}\right) = -q. \end{aligned}$$

N. B. The preceding properties suppose that the equation has been reduced to the form

$$x^2 + px - q = 0;$$

that is, (1), that it has been divided by the coefficient of x^2 ; (2), that all the terms have been transposed and arranged in the first member.

99. *Discussion.* Let us take the general equation

$$x^2 + px = q;$$

which, being resolved, gives

$$x = -\frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}}.$$

In order that the value of this expression, which contains a radical, may be found either exactly or by approximation, the quantity under the radical sign, that is, $q + \frac{p^2}{4}$, must be positive (85). Now $\frac{p^2}{4}$ being necessarily positive, whatever be the sign of p , the sign of $q + \frac{p^2}{4}$ depends principally on that of q , or on that of the known term.

In the first place let q be positive, in which case the equation is

$$x^2 \pm px = +q$$

(distinguishing the signs of the coefficients), whence

$$x = \mp \frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}}.$$

Now it is manifest that the two values of x are real and can be determined exactly, if $q + \frac{p^2}{4}$ be a perfect square; or, if otherwise, to any degree of approximation we please.

The first of these two values is always positive, and is a direct answer to the equation, or to the problem of which the equation is the algebraical translation; for the radical

$$\sqrt{q + \frac{p^2}{4}}$$

being numerically greater than $\frac{p}{2}$, the expression

$$\mp \frac{p}{2} + \sqrt{q + \frac{p^2}{4}}$$

has necessarily the same sign as the radical.

The other value is, for the same reason, negative, because it must have the same sign as that of the radical. Considered independently of its sign, it answers no longer to the equation which has been established, but to one arising from the substitution of $-x$ for $+x$ in that equation; that is, to

$$x^2 \mp px = q.$$

For the solution of this last equation gives

$$x = \pm \frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}},$$

which differs from the former only in its sign.

It is also remarkable that the same equation connects two questions whose enunciations differ only in the meaning of certain conditions. (See the two problems of article 96.)

100. Let q be negative, in which case the equation is of the form

$$x^2 \pm px = -q,$$

and the values of x are

$$x = \mp \frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}.$$

In order that the extraction of the root may be performed, it is necessary that $q < \frac{p^2}{4}$. This condition being satisfied, the two values will be real.

Moreover, since

$$\sqrt{\frac{p^2}{4} - q}$$

is numerically less than $\frac{p}{2}$, the two values are both negative, when p is positive, or when the equation is of the form

$$x^2 + px = -q,$$

and both positive when p is negative, or when the equation is of the form

$$x^2 - px = -q.$$

The same consequences may be deduced from the two properties demonstrated above, that in every equation of the second degree,

$$x^2 + px - q = 0,$$

the algebraic sum of the two roots is equal to the coefficient of the second term taken with the contrary sign; and that their product is equal to the last, or known term, when transposed to the first member of the equation.

For, let q be positive in the second term, and therefore negative in the first; the product of the two roots is therefore negative, and they have contrary signs. Also, their sum is positive or negative, according as p is negative or positive; that is, the root which is numerically the greater of the two will always have the contrary sign to the coefficient p .

But if q be negative in the second member, and consequently positive in the first, the product of the two roots is positive; they have therefore the same sign, that is, are both negative if p be positive, or both positive if p be negative, since in the first case their algebraic sum is negative, and in the second positive.

We can also show, *a priori*, that whenever q is negative in the second member, and p negative in the first, the problem admits of two direct solutions, provided there exist between p and q the relation

$$q < \frac{p^2}{4}.$$

The equation

$$x^2 - px = -q,$$

by changing the signs, may be put under the form

$$px - x^2 = q, \text{ or } x(p - x) = q,$$

which is evidently the algebraical translation of the following problem: *To divide a number p into two parts, of which the product is a given number q* ; for if x be one of these parts, $p - x$ is the other, and their product is $x(p - x)$.

To prove that this problem admits of two direct solutions, we remark that the equation is the same whether x be supposed to represent the greater or the less of the two parts; the equation cannot therefore give one more than the other; it ought therefore to give them both at the same time.

Also, the two parts sought should be such that their sum may be equal to p , and their product to q . Now these same relations exist between the roots of the equation

$$x^2 - px = -q, \text{ or } x^2 - px + q = 0$$

and its coefficients; hence the parts required are equal to the two roots of this equation.

In the second place, in order that the solution of this problem may be possible, we must have $q < \frac{p^2}{4}$.

For whatever be the two parts sought, we may always designate the difference of the two parts by d ; since their sum is p , we have, by the theorem of article 4,

for the greater part $\frac{p}{2} + \frac{d}{2},$

and for the less $\frac{p}{2} - \frac{d}{2}.$

The product of these two is $\frac{p^2}{4} - \frac{d^2}{4}$, which is necessarily less than $\frac{p^2}{4}$, unless we suppose the two parts equal, in which case $d = 0$, and the product is reduced to $\frac{p^2}{4}$.

It is therefore absurd to require that the product, which we have denoted by q , should be greater than $\frac{p^2}{4}$. Whence we may conclude, that whenever the known quantity in the second member is negative, but numerically greater than the square of half the coefficient of the second term, the proposed question is impossible.

Remark. It follows that the greatest product which can be obtained by decomposing a number into two parts, and multiplying these parts together, is the square of half the number. For this product may be expressed by $\frac{p^2}{4} - \frac{d^2}{4}$, which is less than $\frac{p^2}{4}$, but which becomes equal to it if we suppose that $d = 0$, or that the two parts are equal.

Examination of some particular cases.

101. (1.) If, when q is negative, that is, when the equation is of the form

$$x^2 + px = -q$$

(p being of either sign), we suppose $q = \frac{p^2}{4}$, the radical

$$\sqrt{\frac{p^2}{4} - q}$$

becomes nothing, and the values are both reduced to $x = -\frac{p}{2}$; in this case we say that the two roots are equal.

For if we go back to the original equation, and substitute $\frac{p^2}{4}$ for q , it becomes

$$x^2 + px = -\frac{p^2}{4};$$

whence

$$x^2 + px + \frac{p^2}{4} = 0, \text{ or } \left(x + \frac{p}{2}\right)^2 = 0.$$

The first member of this equation is the product of two equal factors. We may say, therefore, that the two roots of the equation are equal, because both factors, when made equal to 0, give the same value for x .

(2.) If, in the equation $x^2 + px = q$, we suppose $q = 0$, the values of x are reduced to

$$x = -\frac{p}{2} + \frac{p}{2}, \text{ or } x = 0,$$

and

$$x = -\frac{p}{2} - \frac{p}{2} = -p.$$

The equation is then $x^2 + px = 0$, or $x(x + p) = 0$; which may be verified either by making $x = 0$, or $x + p = 0$; whence $x = -p$.

(3.) If, in the general equation $x^2 + px = q$, we suppose $p = 0$, there results the equation $x^2 = q$, or $x = \pm \sqrt{q}$; that is, in this case, the two values of x are equal, and of contrary signs; real if q be positive, and imaginary, if it be negative. The equation belongs to the class of equations of two terms treated of in article 92.

(4.) Let us suppose at the same time $p = 0$, and $q = 0$; the equation is reduced to $x^2 = 0$, which gives two values of x , each equal to nothing.

102. It remains to examine a peculiar case which often occurs in the solution of equations of the second degree.

For this purpose let us resume the original equation $ax^2 + bx = c$, which being resolved gives

$$x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}.$$

Suppose that, according to some particular hypothesis made in the given quantities of the problem, $a = 0$; the expression for x becomes

$$x = \frac{-b \pm b}{0},$$

whence

$$x = \frac{0}{0}, \text{ or } x = \frac{2b}{0};$$

the second value takes the form of an infinite quantity, and may be regarded as an answer, if the question admit of infinite solutions (71).

The first result $\frac{0}{0}$ remains to be interpreted.

The hypothesis $a = 0$ reduces the original equation to $bx = c$; whence $x = \frac{c}{b}$, a finite and determinate expression, which must be regarded as the true value of $\frac{0}{0}$ in this case.

To leave no doubt on this subject, let us take the original equation

$$ax^2 + bx = c, \text{ and put } x = \frac{1}{y}.$$

Then the equation becomes

$$\frac{a}{y^2} + \frac{b}{y} = c, \text{ or } cy^2 - by - a = 0.$$

Let $a = 0$; this last equation becomes $cy^2 - by = 0$, which gives $y = 0$, and $y = \frac{b}{c}$.

Substitute these values in the equation $x = \frac{1}{y}$; we deduce from it

$$x = \frac{1}{0} \text{ and } x = \frac{c}{b}.$$

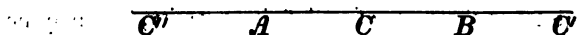
If we suppose $b = 0$, as well as $a = 0$, the value $\frac{c}{b}$ itself becomes $\frac{c}{0}$, or infinite.

For the equation $cy^2 - by - a = 0$, is reduced, by this hypothesis, to $cy^2 = 0$, the two roots of which are each equal to 0. Therefore the corresponding values of x are both infinite.

If, at the same time, $a = 0$, $b = 0$, and $c = 0$, the proposed equation is altogether indeterminate. This is the only case in which the equation of the second degree is so.

We proceed to apply the general discussion to a problem containing all the circumstances which are ordinarily met with in the solution of problems of the second degree.

103. *Two luminaries of different degrees of brightness are placed at the points A and B. At what point of the straight line AB will the quantity of light received from the two bodies be equal.*



(We take for granted this principle of natural philosophy, that the brightness of the same light, as seen from two different distances, varies in the inverse ratio of the squares of the distances.)

Let a be the distance AB of the two lights, b the brightness of the light A at the distance of a unit, and c that of B at the same distance. Let C be the required point. Let $AC = x$; then $BC = a - x$.

Since the brightness of the light A , at the distance unity, is b , by the principle of natural philosophy stated above, its brightness at the distances 2, 3, 4, &c. must be

$$\frac{b}{4}, \frac{b}{9}, \frac{b}{16}, \text{ \&c.};$$

and at the distance x , it must be $\frac{b}{x^2}$. For similar reasons, the brightness of B at the distance $a - x$ is

$$\frac{c}{(a-x)^2};$$

but, from the enunciation, the first degree of brightness is the same as the second; therefore

$$\frac{b}{x^2} = \frac{c}{(a-x)^2};$$

which, when developed and reduced, is

$$(b-c)x^2 - 2abx = -a^2b.$$

This equation gives

$$x = \frac{ab}{b-c} \pm \sqrt{\frac{a^2b^2}{(b-c)^2} - \frac{a^2b}{b-c}}, \text{ or } x = \frac{a(b \pm \sqrt{bc})}{b-c}.$$

The expression may be simplified, since $b \pm \sqrt{bc}$ may be put under the form

$$\sqrt{b} \cdot \sqrt{b} \pm \sqrt{b} \cdot \sqrt{c}, \text{ or } \sqrt{b}(\sqrt{b} \pm \sqrt{c}),$$

and

$$b-c = (\sqrt{b})^2 - (\sqrt{c})^2 = (\sqrt{b} - \sqrt{c})(\sqrt{b} + \sqrt{c}).$$

The root which has the positive sign before the radical gives

$$x = \frac{a\sqrt{b}(\sqrt{b} + \sqrt{c})}{(\sqrt{b} + \sqrt{c})(\sqrt{b} - \sqrt{c})} = \frac{a\sqrt{b}}{\sqrt{b} - \sqrt{c}}.$$

In like manner, the second value of x is

$$x = \frac{a\sqrt{b}(\sqrt{b} - \sqrt{c})}{(\sqrt{b} + \sqrt{c})(\sqrt{b} - \sqrt{c})} = \frac{a\sqrt{b}}{\sqrt{b} + \sqrt{c}}.$$

These simplified values may be obtained immediately from the proposed equations. For, from the equation

$$\frac{b}{x^2} = \frac{c}{(a-x)^2}, \text{ may be deduced } \frac{(a-x)^2}{x^2} = \frac{c}{b};$$

extract the square root of both members, and we have

$$\frac{a-x}{x} = \pm \sqrt{\frac{c}{b}} = \pm \frac{\sqrt{c}}{\sqrt{b}};$$

which, cleared of fractions, becomes

$$a\sqrt{b} - x\sqrt{b} = \pm x\sqrt{c}, \text{ or } x = \frac{a\sqrt{b}}{\sqrt{b} \pm \sqrt{c}}.$$

N. B. The first values were the most complicated, because the equation of the second degree was solved by the general method, which is not so simple as the one which we have just followed.

We have therefore

$$\left. \begin{array}{l} (1.) \quad x = \frac{a\sqrt{b}}{\sqrt{b} + \sqrt{c}} \\ (2.) \quad x = \frac{a\sqrt{b}}{\sqrt{b} - \sqrt{c}} \end{array} \right\} \text{whence} \quad \left\{ \begin{array}{l} a - x = \frac{a\sqrt{c}}{\sqrt{b} + \sqrt{c}}, \\ a - x = \frac{-a\sqrt{c}}{\sqrt{b} - \sqrt{c}}. \end{array} \right.$$

Let $b > c$.

The first value of x , $\frac{a\sqrt{b}}{\sqrt{b} + \sqrt{c}}$, is positive, and less than a , because $\frac{\sqrt{b}}{\sqrt{b} + \sqrt{c}}$ is a fraction less than unity; this value, therefore, gives, for the point required, a point C , between A and B . Also, this point is nearer to B than A ; for, since $b > c$,

$$\sqrt{b} + \sqrt{b}, \text{ or } 2\sqrt{b} > \sqrt{b} + \sqrt{c}, \text{ or } \frac{\sqrt{b}}{\sqrt{b} + \sqrt{c}} > \frac{1}{2},$$

therefore

$$\frac{a\sqrt{b}}{\sqrt{b} + \sqrt{c}} > \frac{a}{2}.$$

This must be the case, as the brightness of A has been supposed greater than that of B .

The corresponding value of $a - x$, namely,

$$\frac{a\sqrt{c}}{\sqrt{b} + \sqrt{c}},$$

is positive, and less than $\frac{a}{2}$, which may be easily verified.

The second value of x ,

$$\frac{a\sqrt{b}}{\sqrt{b} - \sqrt{c}},$$

is positive, and greater than a ; for

$$\frac{\sqrt{b}}{\sqrt{b} - \sqrt{c}} > 1.$$

This second value gives a point C' , situated in the prolongation of AB to the right of the two lights. We may see that, since the light spreads itself equally in all directions, there must be another point in the prolongation of AB , equally illuminated by each light;

and that this point must be nearer to the weaker light than to the other.

We may see why these two values are connected by the same equation. If we take the unknown quantity x to represent AC' instead of AC , we have $BC' = x - a$; and the equation of the problem is

$$\frac{b}{x^2} = \frac{c}{(x - a)^2};$$

and since $(x - a)^2$ and $(a - x)^2$ are identically the same, this is the same equation as the one already established, which must therefore give AC' as well as AC .

The second value of $a - x$, namely,

$$\frac{-a\sqrt{c}}{\sqrt{b} - \sqrt{c}},$$

is negative, and necessarily so, since $x > a$; but, if the signs of the equation

$$a - x = \frac{-a\sqrt{c}}{\sqrt{b} - \sqrt{c}}$$

be changed, it becomes

$$x - a = \frac{a\sqrt{c}}{\sqrt{b} - \sqrt{c}};$$

and this value of $x - a$ represents the absolute value of BC' .

Let $b < c$.

The first value of x , namely,

$$\frac{a\sqrt{b}}{\sqrt{b} + \sqrt{c}},$$

is positive, but less than $\frac{a}{2}$, since

$$\sqrt{b} + \sqrt{c} > \sqrt{b} + \sqrt{b} > 2\sqrt{b}.$$

The corresponding value of $a - x$, namely,

$$\frac{a\sqrt{c}}{\sqrt{b} + \sqrt{c}},$$

is positive, and greater than $\frac{a}{2}$.

On this hypothesis the point C is situated between A and B , but nearer to A than to B .

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The SECOND VALUE of x ,

$$\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}, \text{ or } -\frac{a\sqrt{b}}{\sqrt{c}-\sqrt{b}},$$

is negative. To explain this, we return to the equation, which, when $-x$ is substituted for x , becomes,

$$\frac{b}{x^2} = \frac{c}{(a+x)^2}.$$

Since, before this change, $a-x$ represented the distance of the required point from B , that distance must now be expressed by $a+x$, and the required point is to the left of A ; as, for example, at C'' . Since the brightness of the light B is, by hypothesis, greater than that of A , the second of the equally illuminated points must be nearer to A than B .

The corresponding value of $a-x$,

$$\frac{a-\sqrt{c}}{\sqrt{b}-\sqrt{c}}, \text{ or } \frac{a\sqrt{c}}{\sqrt{c}-\sqrt{b}},$$

is positive; which it must be, since, when x is negative, $a-x$ really is an *arithmetical sum*.

Let $b = c$.

Each of the two first values of x and $a-x$ becomes $\frac{a}{2}$; so that the point which is equally enlightened by both is the middle of AB . This also follows from the hypothesis.

The two other values are reduced to $\frac{a\sqrt{b}}{0}$, or become *infinite*; that is, the second point, which is equally illuminated by both, is at a distance greater than any assignable quantity. This result is perfectly conformable to the hypothesis; for, if we suppose the difference $b-c$, instead of being equal to nothing, to be extremely small, there is a second point, but at a very great distance; which is indicated by the expression for x ,

$$\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}},$$

whose denominator is very small when compared with its numerator. If then $b = c$, or $\sqrt{b} - \sqrt{c} = 0$, the required point can no longer exist, or is at an *infinite* distance.

While on this subject, we may observe, that if, when $b = c$, we take the values of x , which are not simplified, viz.

$$x = \frac{a(b + \sqrt{b}c)}{b - c}, \quad x = \frac{a(b - \sqrt{b}c)}{b - c}.$$

The first, which is the same as

$$x = \frac{a\sqrt{b}}{\sqrt{b} - \sqrt{c}}, \text{ becomes } \frac{2ab}{0},$$

and the second, which corresponds to

$$x = \frac{a\sqrt{b}}{\sqrt{b} + \sqrt{c}}, \text{ becomes } \frac{0}{0}.$$

But $\frac{0}{0}$ is obtained from the common factor $\sqrt{b} - \sqrt{c}$, which is in both terms of this value of x . (See art. 72.)

Both terms of the first value also contain the common factor $\sqrt{b} + \sqrt{c}$; the suppression of which gives

$$x = \frac{a\sqrt{b}}{\sqrt{b} - \sqrt{c}},$$

which becomes

$$x = \frac{a\sqrt{b}}{0} \text{ when } b = c.$$

Let $b = c$, and $a = 0$.

The first values of x and $a - x$ become 0, and the second $\frac{0}{0}$.

This last symbol is here really that of an indeterminate quantity; for the original equation,

$$(b - c)x^2 - 2abx = -a^2b,$$

becomes, on these hypotheses,

$$0.x^2 - 0x = 0;$$

which may be satisfied by placing any number instead of x . Since the two lights are of the same brightness, and placed at the same point, they ought to illuminate every point of the line AB equally.

The solution $x = 0$, given by the first value of x , is one of the infinite number of solutions of which we have spoken.

Let $a = 0$, and b and c be unequal.

Each of the two values of x and $a - x$ becomes 0; which proves, that in this case, the only point equally illuminated by both is the point where the two lights are placed.

The equation is then reduced to $(b - c)x^2 = 0$, and gives the two equal values $x = 0$, $x = 0$.

The preceding discussion is a new instance of the precision with which algebra answers to all the circumstances of the enunciation of a problem.

104. Problem 6. *To find two numbers such that the difference of the products of these numbers by the numbers a and b respectively shall be equal to a given number s , and the difference of their squares shall be equal to a given number q .*

Solution. Let x and y be the numbers sought; we have evidently the two equations

$$\begin{aligned} ax - by &= s, \\ x^2 - y^2 &= q. \end{aligned}$$

From the first, we obtain $x = \frac{by + s}{a}$, a value which, substituted in the second, gives

$$(a^2 - b^2)y^2 - 2bsy = s^2 - a^2q \quad (1);$$

then

$$y = \frac{bs \pm a\sqrt{s^2 - q(a^2 - b^2)}}{a^2 - b^2}.$$

Carrying back this value into the expression for x in terms of y , we find

$$x = \frac{\left(\frac{bs \pm a\sqrt{s^2 - q(a^2 - b^2)}}{a^2 - b^2}\right) + s}{a},$$

whence

$$x = \frac{as \pm b\sqrt{s^2 - q(a^2 - b^2)}}{a^2 - b^2}.$$

(It is necessary to observe that in these values of y and x the two upper signs correspond, as well as the lower.)

Discussion. We shall suppose, in all that follows, that a, b, q, s , are absolute numbers; if it were otherwise, certain terms in the values of x and of y would change their signs, and it would be necessary to make these changes before the discussion.

Let $a > b$, whence $a^2 - b^2$ positive.

First, in order that the two values of x and y may be real, we must have

$$q(a^2 - b^2) < s^2, \text{ whence } q < \frac{s^2}{a^2 - b^2}.$$

Let us suppose this last condition fulfilled, and let us determine the signs by which the two sets of values are effected.

The first set is
$$\begin{cases} x = \frac{as + b\sqrt{s^2 - q(a^2 - b^2)}}{a^2 - b^2}, \\ y = \frac{bs + a\sqrt{s^2 - q(a^2 - b^2)}}{a^2 - b^2}. \end{cases}$$

The two values of this set are necessarily positive, and consequently form a *direct solution* of the problem, such as it has been established.

The second set is
$$\begin{cases} x = \frac{as - b\sqrt{s^2 - q(a^2 - b^2)}}{a^2 - b^2}, \\ y = \frac{bs - a\sqrt{s^2 - q(a^2 - b^2)}}{a^2 - b^2}. \end{cases}$$

The value of x is essentially positive, for from $a > b$ we obtain $as > bs$, and, *a fortiori*,

$$as > b\sqrt{s^2 - q(a^2 - b^2)},$$

since the radical quantity is smaller than s .

As to that of y , it may be positive or negative.

In order that it may be positive, we must have

$$bs > a\sqrt{s^2 - q(a^2 - b^2)};$$

whence raising to the square,

$$b^2 s^2 > a^2 s^2 - a^2 q(a^2 - b^2);$$

or, adding to the two members, $a^2 q(a^2 - b^2)$, and subtracting $b^2 s^2$,

$$a^2 q(a^2 - b^2) > s^2(a^2 - b^2),$$

whence, dividing by $a^2(a^2 - b^2)$,

$$q > \frac{s^2}{a^2}.$$

Thus, in order that the second set may be still a *real and direct solution*, we must have

$$q < \frac{s^2}{a^2 - b^2}, \text{ but } q > \frac{s^2}{a^2},$$

that is, q is comprehended between the two numbers

$$\frac{s^2}{a^2} \text{ and } \frac{s^2}{a^2 - b^2}.$$

(Let us remark, that the condition $q > \frac{s^2}{a^2}$ may be obtained more easily, by going back to the equation for y .

This equation being

$$(a^2 - b^2)y^2 - 2bsy = s^2 - a^2q,$$

we see, that in the hypothesis of $a > b$, it is of the form

$$x^2 - px = -q,$$

if we have

$$a^2q > s^2, \text{ or } q > \frac{s^2}{a^2};$$

and we know (100) that the two roots are then at the same time positive.)

If we had, on the contrary, $q < \frac{s^2}{a^2}$, in which case we should have for a still stronger reason,

$$q < \frac{s^2}{a^2 - b^2},$$

the value of y of the second set would be *negative*; and this set (the sign of y being left out of consideration) would be no longer a *solution* of the problem, as it has been established, but rather of that in which the equations would be

$$\begin{aligned} ax + by &= s, \\ x^2 - y^2 &= q, \end{aligned}$$

and which would differ from the problem only inasmuch as s would express a sum, instead of a difference.

Thus, in the case of $a > b$, the problem admits of two real and direct solutions, while we have

$$q > \frac{s^2}{a^2}, \text{ but } q < \frac{s^2}{a^2 - b^2};$$

and it admits of *one only*, if we have $q < \frac{s^2}{a^2}$.

Taking for a , b , s , any absolute numbers whatever, provided always that $a > b$, and then choosing for q , a number comprehended between the two limits

$$\frac{s^2}{a^2} \text{ and } \frac{s^2}{a^2 - b^2},$$

we shall be certain of obtaining *two direct solutions*.

Let there be, for example, $a = 6$, $b = 4$, $s = 15$, whence we deduce

$$\frac{s^2}{a^2} = \frac{225}{36} = 6\frac{1}{4}, \quad \frac{s^2}{a^2 - b^2} = \frac{225}{20} = 11\frac{1}{4}.$$

We may suppose $q = 10$, for example, and it will become

$$x = \frac{6 \times 15 \pm 4 \sqrt{225 - 20 \times 10}}{20} = \frac{90 \pm 20}{20} = \frac{11}{2} \text{ and } \frac{7}{2},$$

$$y = \frac{4 \times 15 \pm 6 \sqrt{225 - 20 \times 10}}{20} = \frac{60 \pm 30}{20} = \frac{9}{2} \text{ and } \frac{3}{2}.$$

The solutions

$$x = \frac{11}{2}, \quad y = \frac{9}{2} \quad \Big| \quad x = \frac{7}{2}, \quad y = \frac{3}{2},$$

form evidently *two direct solutions* of the equations

$$\begin{aligned} 6x - 4y &= 15, \\ x^2 - y^2 &= 10. \end{aligned}$$

But if we suppose $a = 6$, $b = 4$, $s = 15$, $q = 5$, it would be easy to discover that of the two sets, the first only would give a direct solution.

Particular cases which relate to the Hypothesis of $a > b$.

Let $q = \frac{s^2}{a^2 - b^2}$, whence $q(a^2 - b^2) = s^2$.

The two sets of values of x and y are reduced to

$$x = \frac{as}{a^2 - b^2}, \quad y = \frac{bs}{a^2 - b^2}.$$

So that on this hypothesis, there is only one solution of the problem, and it is *direct*.

Further, let $q = \frac{s^2}{a^2}$; whence $s^2 = a^2 q$, and $s = a\sqrt{q}$.

the first set becomes $\begin{cases} x = \frac{as + b\sqrt{b^2 q}}{a^2 - b^2} = \frac{a^2 + b^2}{a^2 - b^2} \cdot \sqrt{q}, \\ y = \frac{bs + a\sqrt{b^2 q}}{a^2 - b^2} = \frac{2ab}{a^2 - b^2} \cdot \sqrt{q}; \end{cases}$

the second . . . $\begin{cases} x = \frac{as - b\sqrt{b^2 q}}{a^2 - b^2} = \sqrt{q}, \\ y = \frac{bs - a\sqrt{b^2 q}}{a^2 - b^2} = 0. \end{cases}$

And indeed let us suppose $s^2 = a^2 q$ in the equation in terms of y ; it is reduced to

$$(a^2 - b^2) y^2 - 2 b s y = 0,$$

whence we deduce

$$y = 0, y \frac{2 b s}{a^2 - b^2} = \frac{2 a b}{a^2 - b^2} \cdot \sqrt{q}.$$

Let us carry back each of these values into

$$x = \frac{b y + s}{a};$$

there results

$$x = \frac{s}{a} = \sqrt{q}, x = \frac{a^2 + b^2}{a^2 - b^2} \cdot \sqrt{q}.$$

Let there now be $a < b$, whence we have $a^2 - b^2$ negative.

The expressions of x and y may be put under the form

$$x = \frac{-a s \mp b \sqrt{s^2 + p(b^2 - a^2)}}{b^2 - a^2}$$

$$y = \frac{-b s \mp a \sqrt{s^2 + q(b^2 - a^2)}}{b^2 - a^2}.$$

These values are all real, since the quantity under the radical is essentially positive.

As to the signs, the first value of x is essentially negative, and it is the same with the first value of y . So that these values, considered independently of the sign, answer not to the proposed equations, but to the equations

$$b y - a x = s, x^2 - y^2 = q,$$

in the first of which, the order of the difference between the products $a x$ and $b y$ is reversed.

The second value of x is necessarily positive; for from $b > a$, we deduce $b \sqrt{s^2 + q(b^2 - a^2)} > a s$, since the radical quantity is numerically greater than s .

But the second value of y is not always positive. In order that it may be so, we must have the relation

$$a \sqrt{s^2 + q(b^2 - a^2)} > b s,$$

whence, raising to the square

$$a^2 s^2 + a^2 q (b^2 - a^2) > b^2 s^2,$$

or, transposing $a^2 s^2$

$$a^2 q (b^2 - a^2) > (b^2 - a^2) s^2,$$

and, dividing by $a^3 (b^3 - a^3)$,

$$q > \frac{s^3}{a^3}.$$

By giving to a , b , s , q , particular values, such that $b > a$ and $q > \frac{s^3}{a^3}$, the problem will be still susceptible of a *direct solution*.

Finally, let $a = b$; whence we have $a^3 - b^3 = 0$.

The first set of values becomes, on this hypothesis,

$$x = \frac{2as}{0}, \quad y = \frac{2as}{0},$$

and the second,

$$x = \frac{0}{0}, \quad y = \frac{0}{0}.$$

But if we return to the equation

$$(a^3 - b^3) y^3 - 2bsy = s^3 - a^3 q,$$

which, when we make $a = b$, is reduced to

$$-2asy = s^3 - a^3 q,$$

we deduce from it

$$y = \frac{a^3 q - s^3}{2as};$$

and the expression for x in terms of y ,

$$x = \frac{by + s}{a}, \text{ gives } x = \frac{a^3 q + s^3}{2as}.$$

(We should arrive at the same results by imitating the process pursued in article 102; that is, by making, in the equation in terms of y , $y = \frac{1}{z}$.)

In order that the solution

$$x = \frac{a^3 q + s^3}{2as}, \quad y = \frac{a^3 q - s^3}{2as},$$

may be direct, we must have $q > \frac{s^3}{a^3}$.

Of Transformations which may be performed upon Inequalities.

105. In the course of the discussion of the two preceding problems, we have had occasion to make use of several *inequalities*,
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and we have performed upon them transformations analogous to those which we perform upon *equalities*. It is indeed what we are often obliged to do, when in discussing a problem, we wish to establish between the given quantities the necessary relations in order that the problem may be susceptible of a solution direct or at least real, and to fix by the help of these relations, the limits between which the particular values of certain given quantities ought to be found, in order that the enunciation may fall in such or such circumstances. Now, although the principles established for equations, may in general be applicable to inequalities, there are nevertheless some exceptions of which it is necessary to speak, in order to guard beginners against errors which they might commit in making use of the signs of inequality. These exceptions arise from the introduction of *negative expressions*, as *quantities*, into the calculation.

For the sake of clearness, we will go over the different transformations which we may have occasion to perform upon inequalities, taking care to indicate the exceptions of which these transformations are susceptible.

TRANSFORMATION BY ADDITION AND SUBTRACTION. *We may, without any exception, add to the two members of any inequality whatever, or subtract from them, the same quantity; the inequality still subsists in the same sense.*

Thus, let $8 > 3$; we have still

$$8 + 5 > 3 + 5, \text{ and } 8 - 5 > 3 - 5.$$

Let there be also $-3 < -2$; we have still

$$-3 + 6 < -2 + 6, \text{ and } -3 - 6 < -2 - 6.$$

This is evident according to what has been said (63).

This principle enables us, as in equations, to transpose certain terms from one member of the inequality to the other; let there be, for example, the inequality $a^2 + b^2 > 3b^2 - 2a^2$; there results

$$a^2 + 2a^2 > 3b^2 - b^2, \text{ or } 3a^2 > 2b^2.$$

We may, without exception, add member to member, two or more inequalities established in the same sense, and the resulting inequality subsists in the same sense as the proposed inequalities. Thus, from

$$a > b, c > d, e > f,$$

there results

$$a + c + e > b + d + f.$$

But it is not always the same, if we subtract, member by member, two or more inequalities established in the same sense.

Let there be the inequalities $4 < 7$ and $2 < 3$, we have

$$4 - 2, \text{ or } 2 < 7 - 3, \text{ or } 4.$$

But let there be the inequalities $9 < 10$ and $6 < 8$; it becomes, by subtraction, $9 - 6$, or $3 > 10 - 8$, or 2.

We ought then to avoid, as much as possible, this transformation, or when we employ it, to assure ourselves in what sense the resulting inequality exists.

TRANSFORMATION BY MULTIPLICATION AND DIVISION. *We may multiply the two members of an inequality by a positive or absolute number, and the resulting inequality subsists in the same sense.*

Thus from $a < b$, we obtain $3a < 3b$; from $-a < -b$, we deduce $-3a < -3b$.

This principle serves to free a quantity from denominators.

If we have the inequality

$$\frac{a^2 - b^2}{2d} > \frac{c^2 - d^2}{3a},$$

we deduce from it, by multiplying the two members by $6ad$,

$$3a(a^2 - b^2) > 2d(c^2 - d^2).$$

The same principle is applicable to division.

But when we multiply or divide the two members of an inequality by a negative quantity, the inequality exists in a contrary sense.

Let there be, for example, $8 > 7$; by multiplying the two members by -3 , we have, on the contrary, $-24 < -21$.

In the same manner,

$$8 > 7 \text{ gives } \frac{8}{-3}, \text{ or } -\frac{8}{3} < \frac{7}{-3}, \text{ or } -\frac{7}{3}.$$

So that, when we multiply or divide the two members of an inequality, by a number expressed algebraically, we must ascertain whether the *multiplier* or the *divisor* is not negative; for, in this last case, the inequality would exist in a contrary sense.

In the problem of article 104, from the inequality

$$aq^2(a^2 - b^2) > s^2(a^2 - b^2),$$

we have been able to deduce $q > \frac{s^2}{a^2}$, by dividing by $a^2(a^2 - b^2)$, because we have supposed $a > b$, or $a^2 - b^2$ positive.

It is not permitted to change the signs of the two members of an inequality, unless we establish the resulting inequality in a contrary sense; for this transformation is evidently reduced to multiplying the two members by -1 .

TRANSFORMATION BY RAISING TO THE SQUARE. *We can raise to the square the two members of an inequality among absolute numbers, and the inequality will subsist in the same sense.*

Thus, from $5 > 3$, we deduce $25 > 9$; from $a + b > c$, we obtain $(a + b)^2 > c^2$.

But if the two members of the inequality are of any signs whatever, we cannot ascertain beforehand in what sense the resulting inequality will subsist.

For example, $-2 < 3$ gives $(-2)^2$, or $4 < 9$; but $-3 > -5$ gives, on the contrary, $(-3)^2$, or $9 < (-5)^2$, or 25 .

We ought then, before raising to the square, to assure ourselves whether the two members may be regarded as absolute numbers.

TRANSFORMATION BY EXTRACTING THE SQUARE ROOT. *We can extract the square root of the two members of an inequality in absolute numbers, and the inequality subsists in the same sense, between the numerical values of these square roots.*

We remark, first, that we cannot propose to extract the square root of the two members of an inequality *except when they are essentially, positive*, for otherwise, we should be led to *imaginary expressions*, which we could not compare.

But if we have $9 < 25$; we deduce from it $\sqrt{9}$, or $3 < \sqrt{25}$, or 5 .

From $a^2 > b^2$, we deduce $a > b$, if a and b express absolute numbers.

In the same manner, the inequality

$$a^2 > (c - b)^2 \text{ gives } a > c - b,$$

if we have supposed c greater than b ; and $a > b - c$, if, on the contrary, b is greater than c .

In a word, when the two members of an inequality are composed of additive and subtractive terms, *we must take care to write, for the square root of each member, a polynomial, in which the subtractions may be possible.*

106. Problem 7. *Two merchants sell different quantities of the same stuff; the second sells three ells more than the first, and both together receive 35 crowns. Had the second sold what the first did,*

he would have received 24 crowns; had the first sold the same as the second, he would have received $12\frac{1}{2}$ crowns. How many ells did each sell?

First Ans. The first sold 15 ells, the second 18.

Second Ans. The first sold 5 ells, the second 8.

Problem 8. *A merchant owes 6240l. payable in 8 months, and 7632l. payable in 9 months. To pay these, he gives a bill of 14,256l. payable in a year. What is the rate of interest?*

Ans. 10,33 per cent.

Problem 9. *A man has 13,000l. which he divides into two parts, and places them out at interest so as to derive the same income from each. Had the first sum been placed at the same rate of interest as the second, it would have yielded 360l. per annum; had the second been placed at the same interest as the first, it would have yielded 490l. per annum. What were the rates of interest?*

Ans. 7 and 6 per cent.

N. B. The equation of this problem may be solved more simply than by the general method.

Problem 10. *The sum of the areas of two rectangles is q , the sum of their bases is a , and p and p' are the areas of two rectangles, having respectively the base of the first, with the altitude of the second, and the base of the second with the altitude of the first. Required the resolution and discussion of this problem.*

The base of the first is $\frac{a [2p + q \pm \sqrt{q^2 - 4pp'}]}{2(p + p' + q)}$.

Problem 11. *Required the solution and discussion of the following problem: To divide each of two numbers a and b into two parts, so that the product of one part of a by one part of b may be a given number p , and the product of the remaining part of a by the remaining part of b may be another given number p' .*

Problem 12. *To find a number such that its square divided by the product of the differences of this number and two given numbers a and b , may be equal to $\frac{p}{q}$; required the resolution and discussion of this problem.*

Questions concerning Maxima and Minima. Properties of Trinomials of the Second Degree.

107. There is a certain class of problems which relate to the theory of equations of the second degree, and which we often meet with in the Application of Algebra to Geometry. The object of these questions is to determine the greatest or least value which the result of certain arithmetical operations, performed upon numbers, is susceptible of.

Let it be proposed, as the first question, to divide a given number $2a$ into two parts, whose product shall be the greatest possible, or a maximum.

Let us designate by x one of the parts, the other will be $2a - x$, and their product, $x(2a - x)$. By giving to x different values, this product will pass through different degrees of magnitude, and it is required to assign to x the value which shall render this product the greatest possible.

Let us designate by y this greatest product, whose value is at present unknown; we shall have, according to the enunciation, the equation

$$x(2a - x) = y.$$

First regarding y as a known quantity, and deducing from this equation the value of x , we find

$$x = a \pm \sqrt{a^2 - y}.$$

Now it is evident from this result, that the two values of x can be real only while we have $y < a^2$, or at most $y = a^2$; whence we may conclude that the greatest value which can be given to y , or the product of the two parts, is a^2 . But if we make $y = a^2$, there hence results $x = a$.

So that, in order to obtain the greatest product, we must divide the given number $2a$ into two equal parts, and the maximum which we obtain is the square of half of the number, a result at which we have arrived by another method (100).

A more simple solution. Let $2x$ be the difference between the two parts; since their sum is already expressed by $2a$, the greatest of these parts will be (4) represented by

$$\frac{2a + 2x}{2}, \text{ or } a + x,$$

the least by $a - x$, and we shall have for the equation,

$$(a + x)(a - x) = y;$$

or, performing the calculations, $a^2 - x^2 = y$; whence

$$x = \pm \sqrt{a^2 - y}.$$

In order that this value of x may be real, it is necessary that y should be at most equal to a^2 ; and by making $y = a^2$; we obtain $x = 0$, which proves *that the two parts ought to be equal*.

This solution has the advantage of leading to an equation of the second degree with two terms.

108. N. B. In the equations

$$x(2a - x) = y \text{ and } (a + x)(a - x) = y,$$

established above, the quantity x is what we call a *variable*, and the expression $x(2a - x)$, or $(a + x)(a - x)$, is called a certain *function* of the variable. This function, represented by y , is itself another *variable*, whose value depends on that which we give to the first. It is for this reason that analysts sometimes designate the former by the name of *independent variable*, while the latter, or y , receives values dependent upon those which we give to x .

Resolving the two equations

$$x(2a - x) = y \text{ and } (a + x)(a - x) = y,$$

with reference to x , we have

$$x = a \pm \sqrt{a^2 - y},$$

and

$$x = \pm \sqrt{a^2 - y};$$

we may in turn regard y as an *independent variable*, and x as a certain *function* of that variable.

109. Let it be proposed, as a second question, to *divide a number $2a$ into two such parts that the sum of the square roots of these two parts may be a maximum*.

Let us call x^2 one of the parts, $2a - x^2$ will be the other part, and the sum of their square roots will have for its expression

$$x + \sqrt{2a - x^2};$$

it is this expression whose maximum is to be determined.

Let us put

$$x + \sqrt{2a - x^2} = y.$$

To resolve this equation, we must free it from the radical. We have first, by transposing the term x to the second member,

$$\sqrt{2a - x^2} = y - x;$$

whence, raising to the square,

$$2a - x^2 = y^2 - 2xy + x^2,$$

or, arranging with reference to x ,

$$2x^2 - 2y = 2a - y^2,$$

an equation whence we deduce

$$x = \frac{y}{2} \pm \sqrt{\frac{y^2}{4} + \frac{2a - y^2}{2}},$$

or, simplifying,

$$x = \frac{y}{2} \pm \frac{1}{2} \sqrt{4a - y^2}.$$

In order that the two values of x may be real, y^2 must be at least equal to $4a$; then $2\sqrt{a}$ is the *greatest value* which y can receive.

But if we make $y = 2\sqrt{a}$, there results $x = \sqrt{a}$, whence we deduce $x^2 = a$, and $2a - x^2 = a$.

Thus the given number $2a$ must be divided into two equal parts, in order that the sum of the square roots of these two parts may be a maximum. This maximum is moreover equal to $2\sqrt{a}$.

For example, let 72 be the proposed number; we have

$$72 = 36 + 36;$$

whence

$$\sqrt{36} + \sqrt{36} = 12;$$

this is the *maximum* of the values which can be obtained as the sum of the square roots of the two parts of 72.

For let us decompose 72 into $64 + 8$; we have $\sqrt{64} = 8$, $\sqrt{8} = 2 + \text{a fraction}$; whence $\sqrt{64} + \sqrt{8} = 10 + \text{a fraction}$. Again, let there be $72 = 49 + 23$; we have

$$\sqrt{49} = 7, \sqrt{23} = 4 + \text{a fraction};$$

then

$$\sqrt{49} + \sqrt{23} = 11 + \text{a fraction}.$$

Let us consider, for the third example, the expression

$$\frac{m^2 x^2 + n^2}{(m^2 - n^2)x},$$

which it is required to render a minimum (m being supposed $> n$).

Let us put

$$\frac{m^2 x^2 + n^2}{(m^2 - n^2)x} = y;$$

whence

$$m^2 x^2 - (m^2 - n^2)y \cdot x = -n^2;$$

we hence deduce

$$x = \frac{(m^2 - n^2)y}{2m^2} \pm \frac{1}{2m^2} \sqrt{(m^2 - n^2)^2 y^2 - 4m^2 n^2}.$$

Now, in order that the two values of x , corresponding to a value of y , may be real, it is evident that $(m^2 - n^2)^2 y^2$, must be at least equal to $4m^2 n^2$, and consequently that y must be at least equal to $\frac{2mn}{m^2 - n^2}$. So that $\frac{2mn}{m^2 - n^2}$ is the *minimum* of the values which the function y can receive.

But if we make $y = \frac{2mn}{m^2 - n^2}$, in the expression for x , the radical disappears, and the value of x becomes

$$x = \frac{m^2 - n^2}{2m^2} \times \frac{2mn}{m^2 - n^2} = \frac{n}{m}.$$

This value, $x = \frac{n}{m}$, is then that which renders the proposed expression a *minimum*.

110. These examples are sufficient to show the steps which must be followed in the resolution of questions of this kind.

After having formed the algebraic expression of the quantity susceptible of becoming either a maximum or a minimum, we make it equal to any letter whatever y . If the equation which is thus obtained, is of the second degree in terms of x , (x designating the variable quantity which enters into the algebraic expression,) we resolve it with reference to x ; then make equal to zero the quantity under the radical, and obtain from this last equation a value of y , which then represents the maximum or minimum sought. Lastly, substituting this value of y in the expression for x , we have the value of this last variable, which will satisfy the enunciation.

N. B. If it should happen that the quantity under the radical remains essentially positive, whatever be the value of y , we should conclude that the proposed expression can pass through all possible states of magnitude; in other words, that it would have *infinity* for a maximum and zero for a minimum.

Let there be, for a new example, the expression

$$\frac{4x^2 + 4x - 3}{6(2x + 1)};$$

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it may be asked whether this expression is susceptible of a *maximum* or of a *minimum*.

Let us put

$$\frac{4x^2 + 4x - 3}{6(2x + 1)} = y.$$

There results from it the equation

$$4x^2 - 4(3y - 1)x = 6y + 3,$$

whence we deduce

$$x = \frac{3y - 1}{2} \pm \frac{1}{2} \sqrt{9y^2 + 4}.$$

Now, whatever value we may give to y , the quantity under the radical will always be positive. So that y , or the proposed expression, can pass through all possible states of magnitude.

In the preceding examples, the quantity under the radical in the value of x , contains only two parts, the one affected by y , or y^2 , and the other a known quantity; and it has been easy to obtain the *maximum* or *minimum* of which the function was susceptible. But it may happen that the quantity under the radical may be a trinomial of the second degree of the form $my^2 + ny + p$. In this case the question becomes more difficult, and to be able to resolve it completely, it is necessary to demonstrate several properties relating to these trinomials.

Properties of Trinomials of the Second Degree.

111. Every algebraical expression which can be reduced to the form $my^2 + ny + p$, is called a *trinomial of the second degree*; where m , n , and p are known quantities with any sign, and y is a *variable quantity*, or one which may be made to pass through various states of magnitude.

Thus, $3y^2 - 5y + 7$, $-9y^2 + 2y + 5$,

$$(a - b + 2c)y^2 + 4b^2y - 2ac^2 + 3a^2b,$$

are trinomials of the second degree in y .

If the trinomial $my^2 + ny + p$ be put equal to nothing, or if

$$my^2 + ny + p = 0,$$

the values of y are

$$y = -\frac{n}{2m} \pm \frac{1}{2m} \sqrt{n^2 - 4mp},$$

and three different hypotheses may be made with respect to the nature of the values of y which have been obtained.

(1.) $n^2 - 4mp$ may be > 0 , or positive; in which case the values of y are real and unequal, and may have any sign.

(2.) $n^2 - 4mp$ may be $= 0$; in which case the roots are real and equal.

(3.) $n^2 - 4mp$ may be < 0 , or negative; and the two roots are imaginary.

In the first case, that is, whenever a trinomial of the second degree is such that when it is put equal to 0, and the resulting equation is solved, the two roots are real and unequal, every quantity (positive or negative) contained between the two roots, gives a result with a contrary sign to that of the coefficient of y^2 ; but every quantity, which is not contained between the two roots, when substituted for y , gives a result with the same sign as the coefficient of y^2 .

Let y' and y'' be the two roots (which we suppose real) of the equation

$$my^2 + ny + p = 0, \text{ or } m(y^2 + \frac{n}{m}y + \frac{p}{m}) = 0.$$

The first member of this equation may (98) be put under the form $m(y - y')(y - y'')$.

Therefore, $my^2 + ny + p = m(y - y')(y - y'')$; and this is true, whatever value is given to y .

Let y' be the least of the two roots, and let $a > y'$ and $< y''$, or let a lie between y' and y'' , then $a - y'$ is > 0 , and $a - y''$ is < 0 , therefore the factors $a - y'$ and $a - y''$ have contrary signs, and their product is negative. Therefore $m(a - y')(a - y'')$, or $ma^2 + na + p$, is of the contrary sign to that of m .

If, on the other hand, y' and y'' be both greater or both less than a , $a - y'$ and $a - y''$ are of the same sign, and their product is positive; consequently

$$m(a - y')(a - y''), \text{ or } ma^2 + na + p,$$

is of the same sign as m .

In the second case, if the two roots be real and equal, every quantity except the root, when substituted for y in the trinomial, gives a result with the same sign as the coefficient of y^2 .

For, since the two roots are equal, $n^2 - 4mp = 0$, whence $p = \frac{n^2}{4m}$; and the trinomial can be put under the form

$$m y^2 + n y + \frac{n^2}{4m}, \text{ or } m \left(y^2 + \frac{n}{m} y + \frac{n^2}{4m^2} \right), \text{ or } m \left(y + \frac{n}{2m} \right)^2.$$

Now, it is evident that, for every value of y , except $-\frac{n}{2m}$, the quantity

$$\left(y + \frac{n}{2m} \right)^2$$

is positive. Therefore,

$$m \left(y + \frac{n}{2m} \right)^2, \text{ or } m y^2 + n y + p$$

is of the same sign as m .

In the third case, where the two roots are imaginary, every real quantity, positive or negative, when substituted for y , gives a result of the same sign as the coefficient of y^2 .

For, since the roots are imaginary,

$$n^2 - 4mp < 0, \text{ or } 4mp > n^2, \text{ or } \frac{p}{m} > \frac{n^2}{4m^2}.$$

Let
$$\frac{p}{m} = \frac{n^2}{4m^2} + k^2,$$

where k^2 is a quantity which must be positive, it follows that

$$\begin{aligned} m y^2 + n y + p, \text{ or } m \left(y^2 + \frac{n}{m} y + \frac{p}{m} \right) \\ = m \left(y^2 + \frac{n}{m} y + \frac{n^2}{4m^2} + k^2 \right) = m \left(y + \frac{n}{2m} \right)^2 + m k^2, \end{aligned}$$

a quantity which always has the same sign as m , whatever value be substituted for y .

112. The second case leads us naturally to a proposition which is of frequent use in analysis.

Whenever a trinomial of the second degree, $m y^2 + n y + p$, is a perfect square, there exists between its coefficients the relation

$$n^2 - 4mp = 0.$$

For, if this trinomial is a perfect square, and of the form

$$(m' y + n')^2,$$

the two roots of the equation $m y^2 + n y + p = 0$, must be equal. But if they are equal, the quantity under the radical, or $n^2 - 4mp$, must be nothing. Therefore, $n^2 - 4mp = 0$.

Reciprocally, if there exist between the coefficients the relation $n^2 - 4mp = 0$, the trinomial is a perfect square; for, from this relation, we deduce

$$p = \frac{p^2}{4m},$$

and

$$m y^2 + n y + p = m y^2 + n y + \frac{n^2}{4m} = \left(y \sqrt{m} + \frac{n}{2\sqrt{m}} \right)^2.$$

113. Let us see the actual use of these properties, in the resolution of questions of *maxima* and *minima*.

Let it be proposed to determine whether, when we have made x a variable, the function $\frac{x^2 - 2x + 21}{6x - 14}$ can pass through all states of magnitude.

Let us put

$$\frac{x^2 - 2x + 21}{6x - 14} = y;$$

whence

$$x^2 - 2(3y + 1)x = -21 - 14y.$$

There results

$$x = 3y + 1 \pm \sqrt{9y^2 - 8y - 20}.$$

In order that x may be real, it is necessary that

$$9y^2 - 8y - 20$$

should be positive. Now, if we make this quantity equal to zero, it becomes

$$y^2 - \frac{8}{9}y - \frac{20}{9} = 0, \text{ whence } y = 2, \text{ and } y = -\frac{1}{9}.$$

These two values of y being real, it follows from the first of the above mentioned properties, that when we give to y values comprehended between 2 and $-\frac{1}{9}$, such as 1, 0, $-\frac{1}{9}$, ... the value of the trinomial will be *negative*, since the coefficient of y^2 is positive. But by giving to y values not comprehended between 2 and $-\frac{1}{9}$, as 3, 4, ... or $-2, -3, -4, \dots$, we shall obtain a *positive* result. We see then that 2 is, in *absolute numbers*, the *minimum* of the values which y ought to receive, in order that x may be real. If in the above expression of x , we make $y = 2$, the radical disappears, and we find $x = 7$.

Indeed, the expression

$$\frac{x^2 - 2x + 21}{6x - 14}$$

becomes, upon the hypothesis of $x = 7$,

$$\frac{49 - 14 + 21}{42 - 14} = \frac{56}{28} = 2.$$

The root $y = -\frac{1}{3}$ is, in *negative numbers*, the *maximum* of the values which y can receive; and the value of x , corresponding to this *maximum*, is

$$x = 3 \times -\frac{1}{3} + 1 = -\frac{2}{3}.$$

When having expressed x in terms of y , the coefficient of y^2 under the radical sign, is negative, and when the two values of y , deduced from the trinomial made equal to zero, are, the one positive, the other negative, *the positive value is a maximum*, since any greater value would give a result of the same sign as the coefficient of y^2 ; and *the negative value is a minimum* among the negative values which y can receive.

We leave to the learner the care of examining the other circumstances which may be presented; for example, the case in which the coefficient of y^2 being positive, the two values of y are positive; and that in which this same coefficient being positive, the two values are imaginary; and he can moreover exercise himself with the following questions;

To divide a given number $2a$ into two parts, so that the sum of the quotients which are obtained, when mutually divided the one by the other, shall be a minimum.

(Ans. The two parts must be equal, and the *minimum* is 2.)

Let a and b be two given numbers of which a is the greater; it is required that the expression

$$\left(\frac{(x + a)(x - b)}{x^2} \right)$$

should be the greatest possible.

(Ans. *Maximum* $= \frac{(a + b)^2}{4ab}$, and the corresponding value of

x is $\frac{2ab}{a - b}$.)

It is required that

$$\frac{a + x}{x} (b + x)$$

should be the least possible.

(*Minimum* $= (\sqrt{a} + \sqrt{b})^2$; $x = \sqrt{ab}$.)

*On Equations and Problems of the Second Degree, containing
Two or more Unknown Quantities.*

114. Problem 1. *To find two numbers, the sum of whose products by the numbers a and b respectively, is equal to $2s$, and whose product is equal to p .*

Let x and y be the two numbers required; we have the equations

$$\begin{aligned} ax + by &= 2s, \\ xy &= p. \end{aligned}$$

From the first we deduce

$$y = \frac{2s - ax}{b};$$

substitute this value in the second, reduce it, and it becomes

$$ax^2 - 2sx = -bp.$$

Therefore,

$$x = \frac{s}{a} \pm \frac{1}{a} \sqrt{s^2 - abp},$$

and

$$y = \frac{s}{b} \mp \frac{1}{b} \sqrt{s^2 - abp}.$$

This problem has evidently two direct solutions, for

$$s > \sqrt{s^2 - abp};$$

but they cannot be real unless $s^2 >$ or $= abp$.

Let $a = b = 1$; the values of x and y are reduced to

$$x = s \pm \sqrt{s^2 - p}, \text{ and } y = s \mp \sqrt{s^2 - p},$$

in which the values of x are the same as those of y , with the double sign inverted; that is, if $s + \sqrt{s^2 - p}$ be the value of x ,

$$s - \sqrt{s^2 - p}$$

is the corresponding value of y , and the reverse.

This circumstance may be explained, if we observe that the equations become, in this particular case,

$$\begin{aligned} x + y &= 2s, \\ xy &= p; \end{aligned}$$

and the question is reduced to that of finding two numbers whose sum is $2s$, and whose product is p , or in other words, to that of dividing a number $2s$ into two parts whose product is p .

Now, by article 100, these two parts are connected with one another by the same equation of the second degree,

$$x^2 - 2sx + p = 0,$$

the coefficient of whose second term is the sum $2s$, with a contrary sign, and whose last term is the product p of the parts.

115. Problem 2. To find four proportional numbers, the sum of whose extremes is $2s$, that of the means $2s'$, and that of the squares of the extremes and means $4c^2$.

Let u, x, y , and z , be the four numbers; then, by the fundamental properties of proportion, the equations of the problem are

$$u + z = 2s,$$

$$x + y = 2s',$$

$$uz = xy,$$

$$u^2 + x^2 + y^2 + z^2 = 4c^2.$$

At first sight, it may appear difficult to find the values of the unknown quantities; but by means of an *auxiliary unknown quantity*, they may be simply determined.

Let p be the unknown product of the extremes or of the means.

Then

$$(1.) \left\{ \begin{array}{l} u + z = 2s, \\ uz = p, \end{array} \right\} \text{ which give } \left\{ \begin{array}{l} u = s + \sqrt{s^2 - p}, \\ z = s - \sqrt{s^2 - p}. \end{array} \right.$$

$$(2.) \left\{ \begin{array}{l} x + y = 2s', \\ xy = p, \end{array} \right\} \text{ which give } \left\{ \begin{array}{l} x = s' + \sqrt{s'^2 - p}, \\ y = s' - \sqrt{s'^2 - p}. \end{array} \right.$$

(See the preceding problem.)

The determination of the four unknown quantities now depends simply on that of p .

Substitute these values of u, x, y , and z , in the last of the equations of the problem, and we have

$$(s + \sqrt{s^2 - p})^2 + (s - \sqrt{s^2 - p})^2 + (s' + \sqrt{s'^2 - p})^2 + (s' - \sqrt{s'^2 - p})^2 = 4c^2,$$

which, when developed and reduced, becomes

$$4s^2 + 4s'^2 - 4p = 4c^2, \text{ or } s^2 + s'^2 - p = c^2,$$

and

$$p = s^2 + s'^2 - c^2.$$

Substitute this value of p in the expressions for u, x, y , and z , and we have

$$u = s + \sqrt{c^2 - s'^2}, \quad x = s' + \sqrt{c^2 - s^2}, \\ z = s - \sqrt{c^2 - s'^2}, \quad y = s' - \sqrt{c^2 - s^2}.$$

These four numbers are proportional, since

$$uz = (s + \sqrt{c^2 - s'^2})(s - \sqrt{c^2 - s'^2}) = s^2 + s'^2 - c^2,$$

$$xy = (s' + \sqrt{c^2 - s^2})(s' - \sqrt{c^2 - s^2}) = s^2 + s'^2 - c^2.$$

N. B. This problem, which we have taken from the Algebra of Lhuillier, will serve to show how much the introduction of an *un-auxiliary known quantity* into a calculation facilitates the determination of principal unknown quantities. We find, in the work just mentioned, other problems of the same kind, which lead to equations of a degree superior to the second, and which nevertheless may be resolved with the assistance of equations of the first and second degree, by introducing *auxiliary unknown quantities*.

116. Let us now consider the case in which a problem would give rise to any two equations of the second degree with two unknown quantities.

An equation with two unknown quantities is said to be of the *second degree*, when it contains terms in which the *sum of the exponents of the two unknown quantities is equal to 2*, and does not exceed 2. Thus,

$3x^2 - 4x + y^2 - xy - 5y + 6 = 0$, $7xy - 4x + y = 0$,
are equations of the second degree.

It hence follows, that every equation of the second degree with two unknown quantities, is of the form

$$ay^2 + bxy + cx^2 + dy + fx + g = 0,$$

$a, b, c \dots$ representing known quantities, either numerical or algebraic.

Let there be proposed the equations

$$ay^2 + bxy + cx^2 + dy + fx + g = 0,$$

$$a'y^2 + b'xy + c'x^2 + d'y + f'x + g' = 0.$$

We can arrange these two equations with reference to x ; and, they become

$$cx^2 + (by + f)x + ay^2 + dy + g = 0,$$

$$c'x^2 + (b'y + f')x + a'y^2 + d'y + g' = 0.$$

This being laid down, if the two coefficients of x^2 were the same in the two equations, we should obtain, by subtracting these two equations the one from the other, an equation of the first degree in terms of x , which might be substituted for one of the proposed

equations ; from this equation, we should obtain the value of x in terms of y , and carrying back this value into one of the proposed equations, we should arrive at an equation which would contain only the unknown quantity y .

Now, if we multiply the first equation by c' , and the second by c , they become

$$c c' x^2 + (b y + f) c' x + (a y^2 + d y + g) c' = 0,$$

$$c c' x^2 + (b' y + f') c x + (a' y^2 + d' y + g') c = 0,$$

equations which may take the place of the preceding, and in which the coefficient of x^2 is the same.

By subtracting them, we find

$$[(b c' - c b') y + f c' - c f'] x + (a c' - c a') y^2 + (d c' - c d') y + g c' - c g' = 0,$$

an equation which gives

$$x = \frac{(c a' - a c') y^2 + (c d' - d c') y + c g' - g c'}{(b c' - c b') y + f c' - c f'}.$$

This expression for x , substituted in one of the proposed equations, would give a *final equation* in terms of y .

But, without performing this substitution, which would lead to a very complicated result, it is easy to perceive that the equation in terms of y must be, in general, of the fourth degree ; for the numerator of the expression of x being of the form $m y^2 + n y + p$, its square, or the expression of x^2 , is of the fourth degree ; now, this square forms one of the parts of the result of the substitution.

Then, in general, *the resolution of two equations of the second degree with two unknown quantities depends on that of an equation of the fourth degree with one unknown quantity.*

117. There is a class of equations of the fourth degree, the solution of which depends on those of the second degree ; they are contained in the form $x^4 + p x^2 + q = 0$.

To solve this equation let $x^2 = y$, which reduces it to

$$y^2 + p y + q = 0, \text{ or } y = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}.$$

But $x^2 = y$, or $x = \pm \sqrt{y}$.

Therefore,

$$x = \pm \sqrt{-\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}}.$$

We may see, by the resolution of this equation, that the unknown quantity has four values, since each of the signs of the first radical may be combined with each of those of the second; *two of the values are equal respectively to the other two, but with contrary signs.*

Let $x^4 - 25x^2 = -144.$

If we put $x^2 = y$, then $y^2 - 25y = -144$;
whence $y = 16$, and $y = 9.$

Therefore,

(1.) $x^2 = 16$, and $x = \pm 4.$

(2.) $x^2 = 9$, and $x = \pm 3.$

Thus, the four values of x are

$+4, -4, +3, \text{ and } -3.$

Let $x^4 - 7x^2 = 8.$

Put $x^2 = y.$

Then $y^2 - 7y = 8$, $y = 8$, and $y = -1.$

Therefore,

(1.) $x^2 = 8$, or $x = \pm 2\sqrt{2}.$

(2.) $x^2 = -1$, or $x = \pm \sqrt{-1};$

the two last values of x are imaginary.

Let $x^4 - (2bc + 4a^2)x^2 = -b^2c^2.$

Put $x^2 = y.$

Then

$y^2 - (2bc + 4a^2)y = -b^2c^2,$

whence $y = bc + 2a^2 \pm 2a\sqrt{bc + a^2},$

and

$x = \pm \sqrt{bc + 2a^2 \pm 2a\sqrt{bc + a^2}}.$

Extraction of the Square Root of an Expression which is partly Rational and partly Irrational.

118. The solution of a *trinomial equation of the fourth degree* gives rise to a new species of algebraical operation, viz. the extraction of the square root of a quantity of the form $a \pm \sqrt{b}$; a and b being either numerical or algebraical quantities.

Let there be the expression $3 \pm \sqrt{5}$ to be raised to the square, we have

$(3 \pm \sqrt{5})^2 = 9 \pm 6\sqrt{5} + 5 = 14 \pm 6\sqrt{5}.$

Hence

$$\sqrt{14 \pm 6\sqrt{5}} = 3 \pm \sqrt{5},$$

In like manner,

$$(\sqrt{7} \pm \sqrt{11})^2 = 7 \pm 2\sqrt{7} \times 11 + 11 = 18 \pm 2\sqrt{77}.$$

Therefore,

$$\sqrt{18 \pm 2\sqrt{77}} = \sqrt{7} \pm \sqrt{11}.$$

Whence it appears that an expression, such as $\sqrt{a \pm \sqrt{b}}$, may sometimes be reduced to the form $a' \pm \sqrt{b'}$, or $\sqrt{a'} \pm \sqrt{b'}$; and this transformation should be made when it is possible, because in that case there is only one or two simple square roots to be extracted, whereas the expression $\sqrt{a \pm \sqrt{b}}$ requires the extraction of the square root of a square root.

119. *Having given a quantity of the form $a \pm \sqrt{b}$, to discover whether it is the square of an expression of the form $a' \pm \sqrt{b'}$, or $\sqrt{a'} \pm \sqrt{b'}$, and to determine that root.*

Call p and q , the two parts of which the square root of $a + \sqrt{b}$ is composed; p and q are either both irrational, or one is rational and the other irrational.

We must remark, that since

$$\sqrt{a + \sqrt{b}} = p + q, \quad (1)$$

it follows that

$$\sqrt{a - \sqrt{b}} = p - q. \quad (2)$$

Taking the product of these equations, we have

$$\sqrt{a^2 - b} = p^2 - q^2; \quad (3)$$

and, since p and q are either one or both irrationals of the second degree, p^2 and q^2 , and consequently $p^2 - q^2$ are rational quantities; therefore, $\sqrt{a^2 - b}$ is a rational quantity.

Whence we may conclude, that $a \pm \sqrt{b}$ cannot be the square of an expression such as $a' \pm \sqrt{b'}$, or $\sqrt{a'} \pm \sqrt{b'}$, unless $a^2 - b$ is a perfect square.

Let this be the case, and let $\sqrt{a^2 - b} = c$. The equation (3) becomes $p^2 - q^2 = c$.

Equations (1) and (2) squared, give

$$\begin{aligned} p^2 + q^2 + 2pq &= a + \sqrt{b}, \\ p^2 + q^2 - 2pq &= a - \sqrt{b}, \end{aligned}$$

whence, by addition, we obtain

$$p^2 + q^2 = a; \quad (4)$$

but we have also

$$p^2 - q^2 = c. \quad (5)$$

$$\text{Accordingly, } \begin{cases} p = \pm \sqrt{\frac{a+c}{2}}, \\ q = \pm \sqrt{\frac{a-c}{2}}. \end{cases}$$

Therefore,

$$\sqrt{a} + \sqrt{b} = \pm \sqrt{\frac{a+c}{2}} \pm \sqrt{\frac{a-c}{2}},$$

$$\sqrt{a} - \sqrt{b} = \pm \sqrt{\frac{a+c}{2}} \mp \sqrt{\frac{a-c}{2}};$$

or, more distinctly,

$$\sqrt{a} + \sqrt{b} = \pm \left(\sqrt{\frac{a+c}{2}} + \sqrt{\frac{a-c}{2}} \right), \quad (6)$$

$$\sqrt{a} - \sqrt{b} = \pm \left(\sqrt{\frac{a+c}{2}} - \sqrt{\frac{a-c}{2}} \right). \quad (7)$$

These two formulas may be verified *à posteriori*; square both members of the first; then

$$a + \sqrt{b} = \frac{a+c}{2} + \frac{a-c}{2} + 2 \sqrt{\frac{a^2-c^2}{4}} = a + \sqrt{a^2-c^2}.$$

But, since $\sqrt{a^2-b} = c$, $c^2 = a^2 - b$,
and

$$a + \sqrt{b} = a + \sqrt{a^2 - a^2 + b} = a + \sqrt{b}.$$

In a similar way the second formula may be verified.

120. *Remark.* Since the formulas (6) and (7) are true, even when $a^2 - b$ is not a perfect square, they may still be used to give values for the expressions

$$\sqrt{a} + \sqrt{b}, \text{ and } \sqrt{a} - \sqrt{b};$$

but in that case the expressions are not simplified by such a process, since the quantities p and q are of the same form as the given expressions.

It is only proper to make this transformation when $a^2 - b$ is a perfect square.

121. As an example of the use of these formulas, take the numerical expression

$$94 + 42\sqrt{5}, \text{ or } 94 + \sqrt{8820}.$$

Here

$$a = 94, b = 8820,$$

$$c = \sqrt{a^2 - b} = \sqrt{8836 - 8820} = 4,$$

a rational quantity; whence the formula is applicable, and

$$\begin{aligned}\sqrt{94 + \sqrt{8820}} &= \pm \left\{ \sqrt{\frac{94 + 4}{2}} + \sqrt{\frac{94 - 4}{2}} \right\} \\ &= \pm (\sqrt{49} + \sqrt{45}) \\ &= \pm (7 + 3\sqrt{5}),\end{aligned}$$

and $(7 + 3\sqrt{5})^2 = 49 + 45 + 42\sqrt{5} = 94 + 42\sqrt{5}.$

Again, let there be the expression

$$\sqrt{np + 2m^2 - 2m\sqrt{np + m^2}}.$$

Here

$$a = np + 2m^2,$$

$$b = 4m^2(np + m^2),$$

$$c^2 = a^2 - b = n^2p^2,$$

or,

$$c = np,$$

whence the formula is applicable; and the root required is

$$\begin{aligned}\pm \left(\sqrt{\frac{np + 2m^2 + np}{2}} - \sqrt{\frac{np + 2m^2 - np}{2}} \right) \\ = \pm (\sqrt{np + m^2} - m),\end{aligned}$$

and

$$(\sqrt{np + m^2} - m)^2 = np + 2m^2 - 2m\sqrt{np + m^2}.$$

Let it be required to simplify the expression

$$\sqrt{16 + 30\sqrt{-1}} + \sqrt{16 - 30\sqrt{-1}}.$$

By the formulas, we find

$$\sqrt{16 + 30\sqrt{-1}} = 5 + 3\sqrt{-1},$$

$$\sqrt{16 - 30\sqrt{-1}} = 5 - 3\sqrt{-1};$$

therefore

$$\sqrt{16 + 30\sqrt{-1}} + \sqrt{16 - 30\sqrt{-1}} = 10.$$

This last example shows better than all the others the utility of the general problem which we have solved; for it proves that the combination of imaginary expressions may produce *real*, and even *rational* quantities.

The formula may be applied by the student to the following cases,

$$\sqrt{28 + 10\sqrt{3}} = 5 + \sqrt{3};$$

$$\sqrt{1 + 4\sqrt{-3}} = 2 + \sqrt{-3};$$

$$\sqrt{bc} + 2b\sqrt{bc-b^2} + \sqrt{bc-2b\sqrt{bc-b^2}} = \pm 2b;$$

$$\sqrt{ab+4c^2-d^2} + 2\sqrt{4abc^2-abd^2} = \sqrt{ab} + \sqrt{4c^2-d^2}.$$

Indeterminate Analysis of the First and Second Degree.

Introduction.—When the enunciation of a problem furnishes a less number of equations than it has unknown quantities, the problem is called *indeterminate*, since its equations may be satisfied by an infinite number of values attributed to the unknown quantities. But it frequently happens that the nature of the question requires the values of the unknown quantities to be expressed in *whole numbers*; in this case, one of the unknown quantities, to which we may first give a value altogether arbitrary, must receive only entire values, and such that the corresponding value of the other, in each of the other unknown quantities, may be expressed also in entire numbers. Now this condition very much restricts the number of *solutions*, particularly if we consider only *direct solutions*, that is, solutions in entire and positive numbers for all the unknown quantities.

The object of indeterminate analysis of the first degree is to resolve indeterminate questions of the first degree in entire and positive numbers. We shall see hereafter the purpose of indeterminate analysis of the second degree.

I. Equations and Problems of the First Degree with Two Unknown Quantities.

122. Every equation of the first degree with two unknown quantities may be reduced to the form $ax + by = c$; a, b, c , designating entire numbers positive or negative.

We begin by observing, that if the coefficients a and b have a common factor which does not divide the second member c , the equation cannot be satisfied by entire numbers.

For let $a = ha'$, $b = hb'$ the equation becomes

$$ha'x + hb'y = c,$$

whence we deduce

$$a'x + b'y = \frac{c}{h},$$

an equation which cannot be satisfied by any set of entire values of x and y so long as c is not divisible by h .

We suppose in all that follows, that a and b are numbers prime to each other. Since if they have a common factor, c must likewise contain this factor, in which case, it might be suppressed in the equation.

123. For the sake of clearness we will first treat of particular equations and afterwards generalize.

Question 1. *To divide 159 into two parts, one of which shall be divisible by 8 and the other by 13.*

Let us designate by x and y the quotients of the division of the two parts sought by the numbers 8 and 13 respectively; then $8x$ and $13y$ will express the two parts, and we shall have the equation

$$8x + 13y = 159 \quad (1),$$

which, according to the enunciation, is to be resolved by entire and positive numbers for x and y .

We deduce, in the first place, from this equation

$$x = \frac{159 - 13y}{8},$$

or, performing the division as far as possible,

$$x = 19 - y + \frac{7 - 5y}{8}.$$

Now, we perceive that the value of x will be entire if we give to y such a value that $\frac{7 - 5y}{8}$ shall be a whole number; moreover this condition is necessary; so that it is only required that $\frac{7 - 5y}{8}$ should be equal to some whole number. Let t be this whole number (t is called an indeterminate), we shall then have

$$\frac{7 - 5y}{8} = t, \text{ whence } 5y + 8t = 7 \quad (2),$$

and the value of x becomes

$$x = 19 - y + t.$$

Every entire value of t which, substituted in equation (2), will give a similar value for y , will satisfy the condition that $\frac{7 - 5y}{8}$ should be a whole number. Thus the two corresponding values of x and y will be entire, and moreover will satisfy equation (1),

which evidently results from the elimination of t from the two equations

$$\frac{7-5y}{8} = t, \text{ and } x = 19 - y + t.$$

The question is then reduced to resolving by entire numbers equation (2), the coefficients of which are more simple than those of equation (1).

From equation (2) we deduce

$$y = \frac{7-8t}{5},$$

or, performing the division partially,

$$y = 1 - t + \frac{2-3t}{5}.$$

Every entire value of t , which makes $2-3t$ a multiple of 5, will also give for y an entire number, and consequently is suited to the condition. Let us then put $\frac{2-3t}{5} = t'$, t' being a new indeterminate. The equation becomes

$$3t + 5t' = 2 \quad (3),$$

and the value of y is reduced to

$$y = 1 - t + t'.$$

[Equation (2) results moreover from the elimination of t' from the two last equations.]

The question is then reduced to resolving in entire numbers equation (3), from which we deduce

$$t = \frac{2-5t'}{3} = -t' + \frac{2-2t'}{3};$$

let us put

$$\frac{2-2t'}{3} = t'',$$

and there results $2t' + 3t'' = 2 \quad (4),$

and $t = -t' + t''.$

From equation (4) we deduce

$$t' = \frac{2-3t''}{2} = 1 - t'' - \frac{t''}{2}.$$

Lastly, let us make $\frac{t''}{2} = t'''$, and there results $t'' = 2t''' \quad (5),$

and $t' = 1 - t'' - t'''.$

As in equation (5), the coefficient of t''' is unity, it follows that every entire value given to t''' will give likewise an entire value to t'' . Moreover the two principal unknown quantities x and y , and the intermediate unknown quantities t , t' , t'' , and t''' have among themselves the relations expressed by the five equations,

$$\begin{aligned}x &= 19 - y + t, \\y &= 1 - t + t', \\t &= -t' + t'', \\t' &= 1 - t'' - t''', \\t'' &= 2t'''.\end{aligned}$$

So that by giving to t''' any entire value whatever, and going back from the last of these equations to the first two, we shall obtain for x and y corresponding values in whole numbers, which necessarily verify the proposed equation; for, according to the reasoning which we have pursued above, this equation results from the elimination of t , t' , t'' , t''' , from the five equations which we have just established.

But in order that we may assign to t''' only the values to which entire and positive values of x and y correspond, it is requisite to express x and y in an *immediate function** of the indeterminate or auxiliary unknown quantity t''' , by means of the five equations above.

Now the expression for t' becomes, by substituting for t'' its value in terms of t''' ,

$$t' = 1 - 2t'' - t''', \text{ or } t' = 1 - 3t''';$$

then going back to the expression for t ,

$$t = -t' + t'' = -1 + 3t''' + 2t'''.$$

Therefore $t = -1 + 5t'''$.

We shall have, in like manner,

$$y = 1 - (-1 + 5t''') + 1 - 3t'''.$$

Then $y = 3 - 8t'''$.

And lastly,

$$x = 19 - (3 - 8t''') + (-1 + 5t'''), \text{ or } x = 15 + 13t'''.$$

* We call a *function* of a letter, considered as *variable*, every expression which contains this letter combined with known quantities.

It is easy to show by the elimination of t''' , that these two equations reproduce the proposed equation. Indeed, if we multiply the first equation by 13, and the second by 8, and add together the results, it becomes

$$13y + 8x = 159.$$

Let us make successively $t''' = 0, 1, 2, 3$, or $t''' = -1, -2, -3$; the preceding formulas will give all the values of x and y in whole numbers either positive or negative, which will satisfy the proposed equation; but if the enunciation requires, that we should only consider *entire and positive solutions*, t''' must only receive such values as will render $3 - 8t'''$ and $15 + 13t'''$ positive. Now it is evident, that there are only the values $t''' = 0$, and $t''' = -1$, which satisfy this condition; for every positive value of t''' renders y negative, and every negative value, *numerically* greater than 1, renders x negative.

If we make successively

$$t''' = 0, t''' = -1,$$

there results

$$y = 3, y = 11,$$

$$x = 15, x = 2.$$

In these two sets, $x = 15$ and $y = 3$, $x = 2$ and $y = 11$, are the only values which verify the equation $8x + 13y = 159$.

As to the question the conditions of which are expressed algebraically by this equation, since $8x$ and $13y$ express the two parts sought, it follows that 8×15 or 120, and 13×3 or 39, form a *first solution*; and 8×2 or 16, and 13×11 or 143 a *second solution*; that is, the number 159 may be divided into the parts $120 + 39$ or $16 + 143$.

124. For a second example let there be the equation

$$17x - 49y = -8 \quad (1).$$

We deduce from it in the first place

$$x = \frac{49y - 8}{17} = 2y + \frac{15y - 8}{17}.$$

In order that there should be an entire value for x corresponding to an entire value of y , it is only necessary that $15y - 8$ should be a multiple of 17. Let then

$$\frac{15y - 8}{17} = t,$$

t being an intermediate unknown quantity ; there results from it

$$15y - 17t = 8 \quad (2)$$

and

$$x = 2y + t.$$

[The elimination of t from the two equations will reproduce equation (1)].

We deduce from equation (2)

$$y = \frac{8 + 17t}{15} = t + \frac{8 + 2t}{15},$$

and the new expression $\frac{8 + 2t}{15}$ must be a whole number.

Putting $\frac{8 + 2t}{15} = t'$, we obtain $2t - 15t' = -8 \quad (3),$

and

$$y = t + t'.$$

Equation (3) gives

$$t = \frac{15t' - 8}{2} = 7t' - 4 + \frac{t'}{2}.$$

Then $\frac{t'}{2}$ must be a whole number.

Let us make $\frac{t'}{2} = t''$, it becomes $t' = 2t''$,

and

$$t = 7t' - 4 + t''.$$

Now in order to express x and y in a function of the indeterminate t'' , let us resume the four equations,

$$\begin{aligned} x &= 2y + t, \\ y &= t + t', \\ t &= 7t' - 4 + t'', \\ t' &= 2t''. \end{aligned}$$

The last but one becomes

$$t = 7 \times 2t'' - 4 + t'', \text{ whence } t = 15t'' - 4;$$

going back to the second, we have

$$y = 15t'' - 4 + 2t'', \text{ whence } y = 17t'' - 4;$$

and finally, the first becomes

$$x = 2(17t'' - 4) + 15t'' - 4, \text{ or } x = 49t'' - 12.$$

These two formulas reproduce the proposed equation by eliminating t'' ; for if we multiply the first by 49, and the second by 17, and subtract the results, it becomes

$$17x - 49y = -204 + 196 = -8.$$

We see, moreover, that by giving to t'' any positive values whatever, we shall obtain positive values for x and y ; but we must not suppose t'' negative.

$$\begin{array}{ll} \text{Let} & t'' = 1, 2, 3, 4 \dots \\ \text{We find} & y = 13, 30, 47, 64 \dots \\ & x = 37, 86, 135, 184 \dots \end{array}$$

The numbers of *entire and positive solutions* of the proposed equation is then infinite; and the smallest set of values is $x = 37$, $y = 13$.

These values verify the equation; for we have

$$17 \times 37 - 49 \times 13 = 629 - 637 = -8.$$

In this example we have dispensed with going over all the reasoning which we made use of in the first example, in order to account for all the operations; but it is easy for beginners to reproduce them by following the transformations step by step.

125. We may thus recapitulate the preceding method;

Let $ax + by = c$ (1), be the equation which it is required to resolve. *Deduce from this equation the value of the unknown quantity which has the smallest coefficient of x for example, and perform the division as far as possible; an expression is obtained for x in terms of y composed of two parts, the one entire, the other fractional, the latter of which we must endeavour to make a whole number. Make this second part equal to a first indeterminate t ; there will result a new equation in terms of y and t , which we may call equation (2), whose coefficients are more simple than those of equation (1); the value of x is then found in an entire function of y and t , and the proposed equation results from the elimination of t from equation (2) and the equation which gives the value of x in terms of y and t . Deduce from equation (2) the value of y and perform the division as far as possible. Make the fractional part equal to a second indeterminate t' ; whence results an equation (3) in terms of t and t' , more simple than the equations (1) and (2). The value of y is thus found expressed in an entire function of t and t' , and the proposed equation results from the elimination of t and t' from equation (3) and the two equations which give x in an entire function of y and t , and afterwards y in an entire function of t and t' .*

Perform the same operations upon equation (3) as upon equations (1) and (2), and continue this series of operations until you

arrive at a last equation between two indeterminates, one of which has unity for its coefficient.

Finally, from this last equation go back to the preceding and seek by successive substitutions to express x and y in a function of the last indeterminate.

You thus obtain two formulas, by means of which, by giving to the remaining indeterminate arbitrary values you will find all the sets of entire values, as well positive as negative, which are capable of verifying the proposed equation $ax + by = c$.

If we seek only entire and positive values of x and y , the two formulas indicate by their composition, *between what limits the value of the last indeterminate must be comprehended*, in order that this condition may be satisfied.

Remark 1. The process which we have indicated above must always lead to a last equation in which the coefficient of one of the indeterminates is equal to *unity*.

For in the first operation we divide the greater coefficient of the two unknown quantities by the less, in the second the less coefficient by the remainder of the first division; in the third the first remainder by the second, and so on; that is, we apply to the two coefficients the process for finding a common divisor. Then, since by hypothesis the two coefficients are prime to each other (122), we shall in the end arrive at a remainder equal to 1, which will serve as a coefficient to the last but one of the *indeterminate* quantities which have been introduced in the course of the calculation.

(2.) When we apply this process to an equation in which the coefficients of the two unknown quantities contain a common factor, which is not found in the second member, but which we did not at first perceive, the course of the calculation makes known the *impossibility* of resolving the question in whole numbers.

Let there be, for example, the equation $49x - 35y = 11$.

(The factor 7 is common to the coefficients of x and y , but does not enter into the second member.)

We deduce from it

$$y = \frac{49x - 11}{35} = x + \frac{14x - 11}{35}.$$

Putting

$$\frac{14x - 11}{35} = t, \text{ whence } y = x + t,$$

we have

$$x = \frac{35t + 11}{14} = 2t + \frac{7t + 11}{14}.$$

Putting

$$\frac{7t + 11}{14} = t', \text{ whence } x = 2t + t',$$

we find

$$t = \frac{14t' - 11}{7} = 2t' - 1 - \frac{4}{7}.$$

This last equation is evidently *impossible in whole numbers* for t and t' , since $\frac{4}{7}$ is a fraction. It is likewise impossible then to obtain a solution of the proposed equation in entire numbers for x and y .

126. The above process is susceptible of several *simplifications* which it is important to introduce in practice.

Let us resume again the equation already considered,

$$17x - 49y = -8;$$

we deduce from this immediately

$$x = \frac{49y - 8}{17}.$$

We observe that 49 is equal to $17 \times 2 + 15$, or rather that 49 equals $17 \times 3 - 2$; then

$$\frac{49y}{17} = 3y - \frac{2y}{17};$$

thus the value of x takes the form

$$x = 3y - \frac{(2y + 8)}{17};$$

and the question is reduced to finding for y a whole number which will make the expression $\frac{2y + 8}{17}$ entire. Now this expression becomes

$$\frac{2(y + 4)}{17};$$

but the two numbers 17 and 2 are *prime to each other*. Thus, in order that $\frac{2(y + 4)}{17}$ may be an entire number, it is only necessary that $y + 4$ should be divisible by 17.

Let us then make $\frac{y + 4}{17} = t$, t being a whole number entirely

arbitrary ; there results

$$y = 17t - 4,$$

and the value of x becomes

$$x = 3y - 2t,$$

or, substituting for y its value in terms of t ,

$$x = 49t - 12.$$

These formulas give all the entire solutions of the proposed question ; for the elimination of t in the two equations, reproduces the equation

$$17x - 49y = -8.$$

By making $t = 1, 2, 3, 4, \dots$ we find the entire and positive values of x and y ; but we cannot make t negative or equal to 0.

We shall thus perceive the importance of the preceding modifications, since, by means of them, we have had occasion for one indeterminate only, in the course of the calculation.

These modifications occur in almost every example ; but we cannot explain them excepting in particular cases ; and for this reason we will further investigate the following questions.

127. Question 2. *It is required to pay 78 francs in pieces of 5 francs and 3 francs, without any other money ?*

Let x be the number of pieces of 5 francs, and y those of 3 francs, we have the equation $5x + 3y = 78$, which admits, in its solution, only entire and positive values for x and y .

This equation, resolved with reference to y , gives

$$y = \frac{78 - 5x}{3},$$

or, by performing the division,

$$y = 26 - x - \frac{2x}{3},$$

or rather

$$y = 26 - 2x + \frac{x}{3}.$$

By considering the first form of the value of y , we see that the value of y corresponding to an entire value of x , cannot be entire unless $\frac{2x}{3}$ is a whole number ; and since 2 is a primé number in relation to 3, it is only necessary that x should be divisible by 3.

Let then

$$x = 3t;$$

there results

$$y = 26 - x - 2t, \text{ or } y = 26 - 5t.$$

If we consider the second value, we shall see also that x must be a multiple of 3, which gives $x = 3t$,

whence there results

$$y = 26 - 2x + t, \text{ or } y = 26 - 5t.$$

These two formulas show that t must be positive, and cannot have a value greater than $\frac{26}{5}$, or $5\frac{1}{5}$.

Let then

$$t = 0, 1, 2, 3, 4, 5;$$

there results $x = 0, 3, 6, 9, 12, 15,$

$$y = 26, 21, 16, 11, 6, 1.$$

Thus, we are enabled to satisfy the question in six different ways, namely, with 6 pieces of 3 francs, without any of 5 francs; with 21 pieces of 3 francs, and 3 pieces of 5 francs; with 16 pieces of 3 francs, and 6 pieces of 5 francs, and so on.

Question 3. *It is required to find a number which, being divided by 39, gives for a remainder 16, and divided by 56, gives for a remainder 27.*

Call x the entire quotient of the division of the number sought by 39, $39x + 16$ is a first expression for this number. Let y be the entire quotient of the division by 56; $56y + 27$ is a second expression for this number.

We have then the equation

$$39x + 16 = 56y + 27,$$

or, by reducing,

$$39x - 56y = 11 \quad (1).$$

We deduce from this

$$x = \frac{56y + 11}{39} = y + \frac{17y + 11}{39},$$

or

$$x = 2y - \frac{(22y - 11)}{39} = 2y - \frac{11(2y - 1)}{39}.$$

(We have here taken the quotient in excess, because this expression presents the factor 11 separately in the numerator of the fraction.)

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As in the expression $\frac{11(2y-1)}{39}$, the factor 11 is prime to 39, in order that this expression may be a whole number, it is only necessary that $2y-1$ should be divisible by 39.

Put $\frac{2y-1}{39} = t,$

there results $2y - 39t = 1 \quad (2),$

and consequently, $x = 2y - 11t.$

Equation (2) gives

$$y = \frac{39t+1}{2} = 19t + \frac{t+1}{2},$$

putting $\frac{t+1}{2} = t',$

we obtain the equation

$$t = 2t' - 1,$$

and consequently,

$$y = 19t + t'.$$

If in this last equation we substitute for t its value in terms of t' , it becomes

$$y = 19(2t' - 1) + t', \text{ whence } y = 39t' - 19.$$

Substituting this value of y and that of t in the expression for x , we find

$$x = 56t' - 27.$$

From inspection of these two formulas it is evident that t' may have any positive value whatever.

Let $t' = 1$, there results

$$y = 39 - 19 = 20, \quad x = 56 - 27 = 29.$$

Substituting the value of x in the expression $39x + 16$, we obtain $39 \cdot 29 + 16$, or 1147 for the smallest number which satisfies the enunciation of the question.

Again let $t' = 2$, we find

$$x = 56 \cdot 2 - 27 = 85;$$

then

$$39x + 16 = 39 \cdot 85 + 16 = 3331,$$

and so on. Moreover, we may verify the enunciation by the two numbers which we have just obtained.

N. B. The artifice, to which we have had recourse in this question, supposes some practice, but we cannot too much recommend the use of it, since it very much abridges the determination of the values of x and y .

128. If we compare the formulas which will give all the sets of values of x and y in the several questions of which we have hitherto treated, with the equations of these problems, we shall easily perceive that they have this common property, *the coefficients of the indeterminate which enter into these formulas are reciprocally the same (except the sign of one of the two) as the coefficients with which the unknown quantities x and y are affected in this equation*; that is, in the value of x , the coefficient of the indeterminate is equal to the coefficient with which y is affected in the equation, and the value of y the coefficient of the indeterminate is equal to the coefficient of x in the equation, taken with the contrary sign, or reciprocally (as it regards the signs of the two coefficients.)

In order to demonstrate this property let us resume the general equation

$$ax + by = c \quad (1),$$

and suppose that after having applied the method, we have arrived at the two formulas,

$$x = mt + A \quad (2),$$

$$y = nt + B \quad (3).$$

We begin by observing that in these formulas the coefficients m and n must be prime to each other; for if they had a common factor, such that, for example,

$$m = m'k, \quad n = n'k,$$

the formulas would become

$$x = m'kt + A,$$

$$y = n'kt + B;$$

and by making $t = \frac{t'}{k}$, we should obtain

$$x = m't' + A, \quad y = n't' + B;$$

whence it would follow that entire values of x and y would correspond to the fractional value $\frac{t'}{k}$ of t , which would be contrary to the nature of the method, which proposes that all the indeterminates introduced in the course of the calculation should receive only entire values.

This being premised, we have seen already that equation (1) must result from the elimination of t in the two equations (2) and (3).

Now in order to perform this elimination, it is sufficient to multiply equation (2) by n , and equation (3) by m , and then to subtract one from the other ; which gives

$$nx - my = nA - mB,$$

an equation which must be identical with the equation

$$ax + by = c,$$

(since m and n , as well as a and b , are prime to each other), and consequently gives

$$n = a, \quad m = -b.$$

As we can subtract equation (3) from equation (2), we have also

$$my - nx = mB - nA;$$

whence, by comparing it with equation (1),

$$n = -a, \quad m = b.$$

Which was to be demonstrated.

Otherwise. As the values (2) and (3) ought to verify equation (1), whatever t may be, we have necessarily

$$a(mt + A) + b(nt + B) = c,$$

or, by developing and arranging with reference to t ,

$$(am + bn)t + aA + bB = c.$$

But since the supposition of $t = 0$ in the formulas (2) and (3), gives $x = A$ and $y = B$, these values must form a particular set ; thus we have separately

$$aA + bB = c;$$

therefore the preceding equality is reduced to

$$(am + bn)t = 0.$$

Now, in order that this equality may be satisfied for every *entire* value given to t , we must have

$$am + bn = 0; \text{ whence } \frac{n}{m} = -\frac{a}{b};$$

and since we have already seen that m and n are prime to each other, as well as a and b , we must have

$$m = -b, \quad n = a;$$

or rather

$$m = b, \quad n = -a.$$

129. We may give, moreover, a demonstration of this property, which shall be altogether independent of the method which we have pursued for obtaining the values of x and y .

Let the proposed equation always be

$$ax + by = c \quad (1);$$

and suppose that by any means whatever we have found

$$x = \alpha \text{ and } y = \epsilon,$$

for a *first solution* in entire numbers (positive or negative); then we say that all the other solutions are contained in the two formulas,

$$\left. \begin{array}{l} y = \epsilon + at \\ x = \alpha - bt \end{array} \right\} \text{ or, rather } \left\{ \begin{array}{l} y = \epsilon - at, \\ x = \alpha + bt; \end{array} \right.$$

t designating a whole number altogether arbitrary.

Indeed, since α and ϵ form a first set of values for x and y in entire numbers, we have the equality

$$a\alpha + b\epsilon = c \quad (2).$$

Subtracting this equality, member by member, from equation (1), we obtain,

$$a(x - \alpha) + b(y - \epsilon) = 0 \quad (3);$$

an equation which may be substituted for the proposed equation.

Now equation (3) becomes

$$x - \alpha = -\frac{b(y - \epsilon)}{a};$$

and in order that the value of x corresponding to an entire value of y may itself become entire, it is only necessary that $b(y - \epsilon)$ should be divisible by a ; but we have seen (122) that the coefficients a and b are prime to each other (otherwise the equation could not be resolved in entire numbers); then it is only necessary that $y - \epsilon$ should be a multiple of a .

Let us then put

$$y - \epsilon = at,$$

and there results

$$x - \alpha = -bt;$$

• and from these two equations we evidently deduce

$$y = \epsilon + at,$$

$$x = \alpha - bt.$$

As from equation (3) we may also deduce

$$y - \epsilon = -\frac{a(x - \alpha)}{b},$$

if we put $x - a = b t$,
 there will result $y - c = -a t$,
 equations which give

$$x = a + b t,$$

$$y = c - a t.$$

It is easy to show that $y = c + a t$, $x = a - b t$ satisfy the proposed equation whatever be the value of t .

For, if we substitute them in this equation, we find

$$a(a - b t + b(c + a t)) = c,$$

or by reducing, $a a + b c = c$,

since a and c give, by hypothesis, a solution of the proposed equation.

150. *Consequence.* If in the formulas

$$y = c + a t,$$

$$x = a - b t,$$

we make successively

$$t = 0, 1, 2, 3, 4, \dots, \text{ and } t = -1, -2, -3 \dots,$$

they become

$$y = c, c + a, c + 2a, c + 3a \dots,$$

$$x = a, a - b, a - 2b, a - 3b \dots,$$

and

$$y = c - a, c - 2a, c - 3a \dots,$$

$$x = a + b, a + 2b, a + 3b \dots$$

Whence we perceive that all the entire solutions, positive or negative of the proposed question, form *two progressions by difference*, of which the ratio is for the values of x , the coefficient with which y is affected in the equation, and for the values of y , the coefficient with which x is affected in the same equation.

131. *ANOTHER METHOD.* It appears from the analysis given in article 129, that all the difficulty in resolving completely the equation $ax + by = c$, consists in finding a *first solution*, since we may afterwards obtain all the others by means of the formulas

$$x = a - b t, y = c + a t.$$

This consideration leads to a second method for resolving an indeterminate equation. It depends upon the elementary properties of continued fractions.

For example, let there be the equation already considered in article (124),

$$17x - 49y = -8.$$

If we convert $\frac{1}{17}$ into a continued fraction, (see note at the end on *continued fractions*), we obtain the fractions

$$0, \frac{1}{1}, \frac{1}{2}, \frac{1}{17}, \frac{1}{17}.$$

Now we know that *the numerator of the difference between two consecutive results is equal to $a + 1$, if the result from which we subtract it, is of a place indicated by an even number; and to $a - 1$, if indicated by an odd number.*

Then since $\frac{1}{17}$ is of a place designated by an odd number, we must have

$$\frac{17}{49} - \frac{8}{23} = \frac{-1}{49 \times 23}; \text{ whence } 17 \times 23 - 49 \times 8 = -1,$$

(an equality which may moreover be immediately verified.)

This being laid down, let us multiply the two members of the equality thus verified, by 8; that is, by the second member of the proposed equation, *taken with the contrary sign*, it becomes

$$\begin{aligned} 17 \times 23 \times 8 - 49 \times 8 \times 8 &= -8, \\ \text{or } 17 \times 184 - 49 \times 64 &= -8, \end{aligned}$$

an equality which is exact, and which does not differ from the proposed equation, excepting that 184 takes the place of x , and 64 that of y ; whence we see that the proposed equation is necessarily satisfied by

$$x = 184 \text{ and } y = 64.$$

This first solution being found, we have (129) for determining the others, the formulas

$$x = 184 + 49t, y = 64 + 17t.$$

If we only want entire and positive values, we must suppose t positive or equal to 0, -1 , -2 , -3 . The supposition $t = -3$, gives $x = 37$, $y = 13$, that is, the lowest set of values found in article 124.

132. In order to generalise the result, suppose the equation to be resolved is

$$ax - by = c \quad (1),$$

a and b being two absolute numbers, but c being either positive or negative.

Let us convert $\frac{a}{b}$, which from its nature is irreducible (122), into a continued fraction, and let us form the consecutive results,

the last is $\frac{a}{b}$, and the last but one may be represented by $\frac{m}{m'}$, which gives the relation

$$a \times m' - b \times m = \pm 1;$$

that is, $+1$, if the result $\frac{a}{b}$ is of an even place, and -1 , if this result is of an odd place.

Suppose, for a moment, that it is of an even place; we have the verified equality

$$a \times m' - b \times m = +1;$$

let us multiply these two members by c , it becomes

$$a \times m' c - b \times m c = c;$$

a result which does not differ from the equation

$$a x - b y = c,$$

except that $m' c$ and $m c$ take the place of x and y ; therefore, $x = m' c$, and $y = m c$, form *one solution* of the equation.

If the result $\frac{a}{b}$ is of an odd place, we have

$$a \times m' - b \times m = -1,$$

whence by multiplying by -1 ,

$$a \times -m' c - b \times -m c = c.$$

Comparing this verified equality with the equation

$$a x - b y = c,$$

we infer that

$$x = -m' c, y = -m c,$$

as a *solution*.

If the equation is of the form

$$a x + b y = c,$$

that is, if the two coefficients a and b have the same sign, it may be modified and written thus,

$$a x - b \times -y = c;$$

then, by forming, as above, the equality

$$a \times m' c - b \times m c = c,$$

or rather

$$a \times -m' c - b \times -m c = c,$$

we may conclude that

$$x = m' c, y = -m c, \text{ or } x = -m' c, \text{ and } y = m c,$$

form the solution of the equation.

Thus, whatever be the proposed equation, we may always, by means of continued fractions, obtain a *first solution* of this equation; and the formulas

$$x = a - b t, \quad y = c + a t,$$

give all the others.

135. Let us apply this method to a new example.

Let the equation to be resolved be

$$29x + 17y = 250.$$

The fraction $\frac{250}{17}$, converted into a continued fraction, gives for the consecutive results

$$\frac{1}{1}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}.$$

Whence we have the verified equality

$$29 \times 7 - 17 \times 12 = -1,$$

(here the result $\frac{1}{4}$ is of an odd place.)

Let us multiply the two members of this equality by -250 , it becomes,

$$29 \times -1750 - 17 \times -3000 = 250;$$

but the proposed equation may be written thus,

$$29 \times x - 17 \times -y = 250.$$

Whence we see that $x = -1750$, $y = 300$, form a *solution*.

The formulas become then

$$x = -1750 - 17t,$$

$$y = 3000 + 29t.$$

If we wish to consider only solutions in whole and positive numbers, we must suppose t negative; thus by changing the sign of t , we have

$$x = -1750 + 17t, \quad y = 3000 - 29t;$$

and it is evident that the values of x and y will be positive only while we have

$$\begin{cases} 17t > 1750, \\ 29t < 3000, \end{cases} \text{ whence } \begin{cases} t > 103\frac{1}{4}, \\ t < 103\frac{4}{7}, \end{cases}$$

or, by performing the division,

$$t > 102\frac{1}{4}, \text{ but } < 103\frac{4}{7}.$$

Then $t = 103$ is the only value of the indeterminate that makes x and y positive.

For $t = 103$ we find $x = 1$, $y = 13$, values which substituted in the equation give

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$$29 \times 1 + 17 \times 13 = 29 + 221 = 250.$$

We see with what precision the preceding method gives all the solutions of the equation.

134. In some circumstances a *first solution* may be obtained without the necessity of reducing $\frac{a}{b}$ to a *continued fraction*.

(1.) If one of the two coefficients a and b is an *exact submultiple* of the known quantity c , the equation gives immediately a first solution.

For example, let the equation be $5x + 3y = 78$; the coefficient 3 divides 78 and gives for a quotient 26.

Then if we put $x = 0$ and $y = 26$, the equation is satisfied, since it becomes

$$5 \times 0 + 3 \times 26 = 78;$$

the other solutions are found in the formulas

$$\begin{aligned} x &= 3t, \\ y &= 26 - 5t. \end{aligned}$$

Let there be the equation

$$12x + 35y = 156.$$

156 is divisible by 12 and gives for a quotient 13, so that $x = 13$, $y = 0$, give a *first set of values*, and we have for the others

$$x = 13 - 35t, \quad y = 12t.$$

(2.) Whenever we perceive by inspection of the equation, that the sum or the difference of the two coefficients a and b , multiplied respectively by two numbers, gives a divisor of the second member, the first solution is obtained immediately.

For example, let the equation be

$$25x - 16y = 12.$$

Since, by making $x = 2$, $y = 3$, we find

$$25 \times 2 - 16 \times 3 = 2,$$

it follows, that by multiplying the two members of this verified equality by 6, the quotient of 12 divided by 2,

$$25 \times 12 - 16 \times 18 = 12.$$

Whence we conclude that $x = 12$, and $y = 18$, satisfy the proposed equation.

Again, let there be the equation

$$18x - 47y = 0;$$

it is evidently satisfied by

$$x = 0, \quad y = 0.$$

So that the general formulas are

$$x = 47t, y = 13t.$$

These means of finding a first solution are peculiar to certain equations, while the conversion into a continued fraction is a method by which we are always sure of arriving at it.

We recommend to beginners to familiarize themselves equally with both these methods for resolving the equation $ax + by = c$.

135. By a mere inspection of the signs of the equation

$$ax + by = c,$$

we can determine whether the number of solutions in *whole and positive numbers*, is limited or infinite.

(1.) Whenever b is positive, (a being always supposed positive,) the number of solutions is *limited*.

For, from this equation we deduce

$$x = \frac{c - by}{a}.$$

This being laid down, if c is negative, whatever positive value is given to y , the corresponding value of x will be negative; so that in this case the equation *admits of no solution*.

If c is positive, we cannot give to y positive values greater than $\frac{c}{b}$, otherwise x would be negative.

(2.) While b is negative, whatever be the sign of c , the number of solutions is *unlimited*.

For, the formulas $x = a - bt$, $y = c + at$, become, when we give to b its sign,

$$x = a + bt, y = c + at.$$

Now by taking the most unfavorable case, that in which a and c are two negative numbers, it is sufficient in order that x and y may be positive, to suppose for t numerical values greater than those of $\frac{a}{b}$ and $\frac{c}{a}$. Thus, we may give to t any entire values whatever which are greater than these two quotients.

In the case where the number of solutions is *limited*, we may always fix the limits between which the values of the indeterminate t ought to be comprehended, by considering the two formulas, which in this case are

$$x = a - bt,$$

$$y = c + at.$$

For this it is sufficient to express the inequalities

$$\begin{aligned} a - b t &> 0, \\ c + a t &> 0, \end{aligned}$$

and to deduce from them, according to the transformations of two other inequalities, explained in article 105, in which the two first members contain only t . We obtain thus the whole number of solutions of which the question is susceptible.

II. Of Equations and Problems with Three or more Unknown Quantities.

136. LET us consider, in the first place, the case of *two equations with three unknown quantities*.

For example, let there be the two equations

$$5x + 4y + z = 272 \quad (1),$$

$$8x + 9y + 3z = 656 \quad (2),$$

in one of which the unknown quantity z is affected by a coefficient equal to unity. We commence by eliminating it.

For this purpose let us multiply the first equation by 3, and subtract the second from the first; we have

$$7x + 3y = 160 \quad (3),$$

an equation which may take the place of equation (2).

Applying to equation (3) the first method, we find the two formulas

$$x = 1 - 3t,$$

$$y = 51 + 7t.$$

Substituting these two expressions for x and y in the first equation, we obtain

$$5(1 - 3t) + 4(51 + 7t) + z = 272,$$

or by reducing

$$z = 63 - 13t.$$

The three unknown quantities are thus found expressed in an *entire function* of the indeterminate t . So that by giving to t any entire values, whatever, we shall obtain similar ones for x , y , and z ; and these values will satisfy the two proposed equations, since according to what we have just said the set of three formulas is *equivalent to two equations*.

If we seek entire and positive values for x , y , and z , it is evident that t cannot be positive, for x would be negative; but we may suppose

$$t = 0, -1, -2, \dots \text{ to } t = -\frac{1}{7}, \text{ or } -7\frac{1}{7}.$$

By making

$$t = 0, -1, -2, -3, -4, -5, -6, -7,$$

we find

$$x = 1, 4, 7, 10, 13, 16, 19, 22;$$

$$y = 51, 44, 37, 30, 23, 16, 9, 2;$$

$$z = 63, 76, 89, 102, 115, 128, 141, 154;$$

whence it will be seen that the problem is susceptible of *eight different solutions*.

Let us verify only the extreme solutions

$$(1.) \ x = 1, y = 51, z = 63, \text{ give } \begin{cases} 5 \cdot 1 + 4 \cdot 51 + 1 \cdot 63 = 272; \\ 8 \cdot 1 + 9 \cdot 51 + 3 \cdot 63 = 656; \end{cases}$$

$$(2.) \ x = 22, y = 2, z = 154, \text{ give } \begin{cases} 5 \cdot 22 + 4 \cdot 2 + 1 \cdot 154 = 272. \\ 8 \cdot 22 + 9 \cdot 2 + 3 \cdot 154 = 656. \end{cases}$$

137. Let there be, for example, the equations

$$6x + 7y + 4z = 122 \quad (1),$$

$$11x + 8y - 6z = 145 \quad (2).$$

In order to eliminate z from these two equations, multiply the first by 3, and the second by 2, then adding the results, member to member, we have

$$40x + 37y = 656 \quad (3),$$

an equation for which we find, according to the first method,

$$x = 37t + 9,$$

$$y = 8 - 40t.$$

Substitute these expressions of x and y in equation (1), and it becomes

$$6(37t + 9) + 7(8 - 40t) + 4z = 122,$$

or, by performing the calculation and reducing

$$2z - 29t = 6 \quad (4).$$

Here the unknown quantity z is not, like x and y , expressed in an entire function of the indeterminate t . So that it is necessary to apply to equation (4) one of the two known methods.

We have for the formulas relative to this equation

$$t = 2t',$$

$$z = 29t' + 3.$$

Since, moreover, every entire value of t , substituted in the expressions of x and y , will give similar values for the unknown quantities, it follows, that if we put $2t'$ in the place of t in these expressions, which give

$$\begin{aligned}x &= 74t' + 9, \\y &= 8 - 80t',\end{aligned}$$

these formulas, combined with

$$z = 29t' + 3,$$

will comprehend all the *sets of entire values* of x , y , and z , which can verify the proposed equation.

If we wish for only *direct* solutions, it is evident that t' cannot be positive, since y would then become negative, and t' cannot be negative, because z and x would then be negative. But the hypothesis $t' = 0$, gives $x = 9$, $y = 8$, $z = 3$; this set then is the only one which can satisfy the two equations.

By reviewing the steps which we have taken, we derive the following general rule; *Eliminate one of the unknown quantities from the proposed equations, and seek for the equation resulting from elimination, the two formulas which give the unknown quantities which enter into them, in an entire function of an indeterminate t . Substitute these expressions in one of the proposed equations, which gives a new equation containing only t and the unknown quantity which we have just eliminated. Determine, for this new equation, the two formulas which give the expression for the two unknown quantities which enter into it, in an entire function of a second indeterminate t' . Lastly, substitute the expression for t in those of the two first unknown quantities. The values of the three unknown quantities are thus expressed in an entire function of t' ; after which it is only required to determine for t' the limits between which these values must be found, in order that the principal unknown quantities may be entire and positive.*

N. B. Whenever one of the unknown quantities has unity for its coefficient in one of the equations, it is more simple to eliminate this unknown quantity, because that after having expressed the two other in an entire function of the same indeterminate, if we substitute these values in the equation in which the third unknown quantity is affected with a coefficient equal to unity, we obtain immediately this third unknown quantity in an entire function of the same indeterminate, so that in this case only one opera-

tion is necessary. The two equations of article 136 afford an example of this.

138. The following are the steps which we must take for three equations with four unknown quantities. *After having eliminated one of the unknown quantities, we express by means of the two equations which result, and according to what we have just laid down, the three other unknown quantities in an entire function of the same indeterminate, and substitute these values in one of the proposed equations. If in the new equation, the coefficients of the two unknown quantities which enter into it, differ from unity, we establish two formulas which give these unknown quantities in an entire function of a second indeterminate, we then substitute in the expressions for the three first unknown quantities the first indeterminate in a function of the second, and we thus obtain the four primitive unknown quantities in an entire function of the second indeterminate.*

The reasoning is the same for four equations with five unknown quantities, and so on. We propose the following questions as an exercise for the learner.

Question 3. *A coiner has three kinds of silver. In 8 ounces or 1 mark, the first contains 7 ounces of pure silver, the second $5\frac{1}{2}$ ounces, and the third $4\frac{1}{2}$ ounces. He wishes to make a mixture of 30 marks in weight, which contains 6 ounces of pure silver for every 8 ounce weight. How many marks (in whole numbers) ought he to take of each sort?*

$$\text{Answer. } \begin{cases} x = 10, 12, 14, 16, 18, \\ y = 20, 15, 10, 5, 0, \\ z = 0, 3, 6, 9, 12, \end{cases}$$

that is, 5 solutions by admitting 0 for the value of y and z .

Question 4. *It is required to find three whole numbers such that the sum of their products by the respective numbers 3, 5, and 7, may be equal to 560, and the sum of their products by the squares of these numbers 9, 25, and 49, may equal 2920.*

$$\text{Answer. } \begin{cases} x = 15, 50, \\ y = 82, 40, \\ z = 15, 30, \end{cases} \text{ that is, two solutions.}$$

Question 5. *It is required to find a number N , which being divided by 11, the remainder is 3, being divided by 19, gives the remainder 5, and divided by 29, gives the remainder 10?*

Answer. $N = 4128 + 6061t$, so that 4128 is the smallest number that satisfies the enunciation.

Question 6. To find such a value of x that the expressions

$$\frac{3x-10}{7}, \frac{11x+8}{17}, \frac{16x-1}{5},$$

may be whole numbers?

Answer. $x = 211 + 595t$, t being an indeterminate.

139. If in the sixth question we designate the quotients

$$\frac{3x-10}{7}, \frac{11x+8}{17}, \frac{16x-1}{5},$$

by y , z , and v , we have for the equations of the problem

$$3x - 10 = 7y; 11x + 8 = 17z; 16x - 1 = 5v;$$

$$\text{or } 3x - 7y = 10; 11x - 17z = -8; 16x - 5v = 1.$$

We must then apply to these equations the process laid down in the preceding article for three equations with four unknown quantities. But we will now proceed to develop another more simple method for determining the value of x , which is here the principal unknown quantity. This method is moreover applicable to all questions of the same kind.

In the first place, if we consider the third expression $\frac{16x-1}{5}$,

it may be reduced to $3x + \frac{x-1}{5}$; so that in order that it should

be entire it is only necessary that $x-1$ should be a multiple of 5.

Let us put

$$\frac{x-1}{5} = t, \text{ there results } x = 1 + 5t.$$

Every entire value of t , substituted in this formula, will give for x a number which will satisfy the third condition of the enunciation.

Let us now substitute this value of x in the first expression $\frac{3x-10}{7}$; it becomes

$$\frac{15t-7}{7}, \text{ or } 2t-1 + \frac{t}{7};$$

whence we see that this new expression will be entire if we suppose $t = 7t'$, moreover this condition is necessary. So that in order that the first and third expressions should be entire, it is necessary that we should have $x = 1 + 5t$, t being of the form $t = 7t'$; which gives $x = 1 + 35t'$.

Let us carry this new value into the second expression

$$\frac{11x+8}{17}; \text{ it becomes } \frac{385t'+19}{17},$$

or

$$23t' + 1 + \frac{2-6t'}{17}.$$

Now we have

$$2 - 6t' = 2(1 - 3t');$$

moreover 2 is prime to 17; then in order that the second member may be a whole number, it is only necessary that $1 - 3t'$ should be divisible by 17.

Putting

$$\frac{1 - 3t'}{17} = t'',$$

we deduce from it

$$t' = \frac{1 - 17t''}{3};$$

or, by performing the division,

$$t' = -6t'' + \frac{t'' + 1}{3}.$$

Let

$$\frac{t'' + 1}{3} = t''',$$

we obtain

$$t' = 3t''' - 1,$$

whence we deduce

$$t' = -6t'' + t''', \text{ or } t' = -17t''' + 6.$$

Carrying this value into the expression

$$x = 1 + 35t',$$

we obtain, all reductions being made,

$$x = 211 - 595t'''.$$

Such is the formula which will give all the values of x capable of satisfying the enunciation.

Let $t''' = 0$, we find $x = 211$; this is the smallest of all the numbers sought. By supposing for t''' any negative, values we shall obtain other solutions.

N. B. We remark that 595, the coefficient of t''' in the formula, is the product $7 \times 17 \times 5$ of the denominators of the three proposed expressions. It will be easy to take account of this property, which is modified when the denominators are not prime to each other; for, in this case, the coefficient is equal to the most simple multiple of the denominators.

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140. It still remains to speak of the problems which are said to be *more than indeterminate*, that is, of problems for which the number of equations is less by *two or more units* than the number of unknown quantities.

In the first place, let there be the equation with three unknown quantities

$$ax + by + cz = d.$$

If we transpose the term cz to the second member, it becomes

$$ax + by = d - cz, \text{ or } ax + by = c',$$

(designating by c' the quantity $d - cz$, which for the present we consider as known.)

This being premised, we establish for the equation

$$ax + by = c',$$

the two formulas $x = a - bt$, $y = b + at$. After this we restore in a and b , for c' its value $d - cz$; then x and y are found expressed in an *entire function of the indeterminate* and of the *third unknown quantity* z .

Let it be proposed, for example, *to pay 187 francs with pieces of 5 francs, 6 francs, and 20 francs, without any other coin.*

Let us designate by x , y , and z , the number of pieces required of each sort; we have the equation

$$5x + 6y + 20z = 187,$$

which is reduced to

$$5x + 6y = 187 - 20z = c'.$$

Deducing from this equation the value of x , we have

$$x = \frac{c' - 6y}{5},$$

or

$$x = -y + \frac{c' - y}{5}.$$

Putting

$$\frac{c' - y}{5} = t,$$

we deduce from it

$$y = c' - 5t,$$

whence

$$x = -c' + 6t.$$

Restoring for c' in these formulas its value $187 - 20z$, we find

$$y = 187 - 20z - 5t,$$

$$x = -187 + 20z + 6t.$$

So long as we admit for x and y whole numbers, positive or negative, we may give to z and t values altogether arbitrary, but if we wish to satisfy the enunciation directly, the same form of the proposed equation

$$5x + 6y + 20z = 187,$$

shows that z cannot receive values greater than $\frac{187}{20}$, or $9\frac{7}{20}$, since otherwise x or y would be negative.

Let us put successively

$$z = 0, 1, 2, 3, \dots, 8, 9.$$

If we make $z = 0$, the values of x and y become

$$x = -187 + 6t,$$

$$y = 187 - 5t;$$

formulas which show that t must be $> \frac{187}{6}$, but $< \frac{187}{5}$, or $> 31\frac{1}{6}$, but $< 37\frac{4}{5}$; then t can have six values, namely,

$$32, 33, 34, 35, 36, \text{ and } 37.$$

Thus, for $z = 0$, we have

$$t = 32, 33, 34, 35, 36, 37,$$

$$x = 5, 11, 17, 23, 29, 35,$$

$$y = 27, 22, 17, 12, 7, 2.$$

Let $z = 1$, we find

$$x = -167 + 6t,$$

$$y = 167 - 5t;$$

whence $t > \frac{167}{6}$, or $27\frac{5}{6}$, but $< \frac{167}{5}$, or $33\frac{4}{5}$, which gives these six values,

$$28, 29, 30, 31, 32, \text{ and } 33.$$

Thus, for $z = 1$, we have

$$t = 28, 29, 30, 31, 32, 33,$$

$$x = 1, 7, 13, 19, 25, 31,$$

$$y = 27, 22, 17, 12, 7, 2.$$

For $z = 2$,

$$t = 25, 26, 27, 28, 29,$$

$$x = 3, 9, 15, 21, 27,$$

$$y = 22, 17, 12, 7, 2.$$

For $z = 3$,

$$t = 22, 23, 24, 25,$$

$$x = 5, 11, 17, 23,$$

$$y = 17, 12, 7, 2.$$

For $z = 8$, the formulas would be

$$\begin{aligned}x &= -27 + 6t, \\y &= 27 - 5t;\end{aligned}$$

whence $t > \frac{27}{6}$ or $4\frac{1}{2}$, but $< \frac{27}{5}$ or $5\frac{1}{5}$. So that t can only receive the value $t = 5$, which gives $x = 3$, $y = 2$.

Finally, from the hypothesis $z = 9$, there is no solution, for the formulas become

$$x = -7 + 6t, \quad y = 7 - 5t,$$

whence $t > \frac{7}{6}$ or $1\frac{1}{6}$, but $t < \frac{7}{5}$ or $1\frac{2}{5}$.

141. It will thus be sufficiently plain what is necessary to be done in the case of two equations with four unknown quantities, and three equations with five unknown quantities. But, we will give a complete resolution of one question of this kind in order to show, how, by the aid of some particular considerations, we may frequently simplify the calculations.

Question 7. *A farmer bought a hundred head of cattle for 100 dollars, namely, oxen at 10 dollars each, cows at 5 dollars, calves at 2 dollars, and sheep at $1\frac{1}{2}$ dollar each. How many of each did he purchase?*

Let x, y, z, u , be the numbers sought, we have the equations

$$\begin{aligned}x + y + z + u &= 100, \\10x + 5y + 2z + \frac{1}{2}u &= 100,\end{aligned}$$

or, reduced,

$$\begin{aligned}x + y + z + u &= 100, \\20x + 10y + 4z + u &= 200.\end{aligned}$$

By subtracting the first equation from the second, we obtain

$$19x + 9y + 3z = 100,$$

an equation which must be treated in the same way as that of the preceding article. But in the first place, we observe that it is preferable to express y and z in an entire function of x . (1.) Because it is evident that x ought not to have values greater than $\frac{100}{19}$ or $5\frac{5}{19}$. (2.) Because the coefficients of y and z have a common factor, which necessarily introduces a condition for determining the proper values of x .

This being premised, let us transpose the term $19x$, it becomes

$$9y + 3z = 100 - 19x,$$

or rather

$$3y + z = \frac{100 - 19x}{3}.$$

Now, since we require for x, y, z, u , positive whole numbers, it is necessary that $\frac{100 - 19x}{3}$ should be whole and positive; but evidently $x = 1$ and $x = 4$, are the only values which can satisfy this twofold condition. So that already we can only have the values $x = 1$ and $x = 4$.

Let $x = 1$, there results

$$3y + z = 27, \text{ or } z = 27 - 3y.$$

Substituting these values of x and z in the first equation proposed, we find $u = 72 + 2y$.

The first of these two formulas shows that y cannot be > 9 ; so that for $x = 1$, we have

$$\begin{aligned} y &= 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \\ z &= 27, 24, 21, 18, 15, 12, 9, 6, 3, 0, \\ u &= 72, 74, 76, 78, 80, 82, 84, 86, 88, 90. \end{aligned}$$

Let $x = 4$, it becomes

$$3y + z = 8, \text{ whence } z = 8 - 3y,$$

and

$$u = 88 + 2y.$$

This expression for z shows that y cannot be > 2 ; so that for $x = 4$, we find

$$\begin{aligned} y &= 0, 1, 2, \\ z &= 8, 5, 2, \\ u &= 88, 90, 92. \end{aligned}$$

Whence we see that the proposed question is susceptible only of thirteen and of ten solutions, if we except the solutions of 0.

III. Of Indeterminate Analysis of the Second Degree.

142. We propose, in this part, as in Indeterminate Analysis of the First Degree, to resolve, by entire numbers, problems, which lead to a number of equations less than the number of unknown quantities. But as, in general, an equation of the second degree, with two unknown quantities, gives one of them in an *irrational function* of the other, it follows that the question consists, (1.) in determining, for one of the unknown quantities, rational values, which have the property of giving similar ones for the second. (2.) To choose among the values of the first unknown quantity the entire values which will give similar ones for the second. We perceive, therefore, that indeterminate analysis of the second degree

must present greater difficulties than that of the first degree. It is indeed one of the most difficult theories of algebraic analysis, and would lead us altogether beyond the elements of the science. We refer the student, for further information, to the *Théorie des Nombres* of Legendre.

We will explain, however, the resolution, in entire numbers, of a series of questions with two unknown quantities, whose equations contain only the *rectangle* or *product of the unknown quantities*, without including any of the two squares.

These questions, which are in themselves very curious, are taken from the Algebra of M. Lhuillier, a work from which we have already taken the enunciations of several problems.

143. Question 1. *It is required to find in entire numbers the sides of a rectangle, whose surface contains four times as many square feet as its perimeter contains feet.*

Let x and y be the sides of the rectangle expressed in feet, xy expresses its surface, and $2x + 2y$ its perimeter.

From the nature of the enunciation, we have the equation

$$xy = 8x + 8y.$$

From this equation we deduce

$$x = \frac{8y}{y-8}, \text{ or } x = 8 + \frac{64}{y-8}.$$

From the form of the value of x and y , it is evident that for an entire value of y , we cannot obtain a similar one for x , except when $y-8$ is a divisor of 64.

Let us suppose then, that we have determined all the divisors of 64, and that we take them with the signs $+$ and $-$, we shall have $y-8=1, 2, 4, 8, 16, 32, 64, | -64, -32, -16, -8, -4, -2, -1$; whence

$y=9, 10, 12, 16, 24, 40, 72, | -56, -24, -8, 0, 4, 6, 7$;

Substituting in the expression for x , instead of $y-8$, its different values, and reducing, we shall find

$x=72, 40, 24, 16, 12, 10, 9, | 7, 6, 4, 0, -8, -24, -56$.

If we look at the last two lines of the calculation, we shall see that the values of x are the same of those of y , but in the inverse order, which ought to be the case, since the equation of the problem does not change when we put x in the place of y , and the reverse. Whence we conclude, leaving out of consideration negative solutions, that the sets of values really different are

$$y = 9, 10, 12, 16,$$

$$x = 72, 40, 24, 16.$$

Thus the question is susceptible of 4 solutions.

Let us verify the set $y = 10, x = 40$.

The base of the rectangle contains 40 feet, and its altitude 10, and its surface is 400 square feet. On the other hand its perimeter is equal to $2(40 + 10)$ or 100 feet; now 400 is quadruple of 100.

Let us generalise this question, and propose to ourselves to determine a rectangle whose surface contains m times as many square yards as its perimeter contains yards.

We have the equation

$$xy = m(2x + 2y) = 2mx + 2my.$$

Whence, deducing the value of x by performing the division,

$$x = 2m + \frac{4m^2}{y - 2m}.$$

We remark, in the first place, that it is useless to take any notice of the negative divisors of $4m^2$; since if $y - 2m$ is negative and numerically smaller than $2m$, y is then positive; but then when $\frac{4m^2}{y - 2m}$ is negative and numerically greater than $2m$, the corresponding value of x is negative. The contrary would take place if $y - 2m$ was negative, and numerically greater than $4m^2$; that is, y would be negative and x would be positive. Now it is only proposed to admit direct solutions of the question.

This being premised, let d, d', d'', d''' , be the divisors of $4m^2$, we have

$$y - 2m = d, d', d'', d''';$$

whence

$$y = 2m + d, 2m + d', 2m + d'', 2m + d''',$$

and consequently, by designating by q, q', q'', q''' , the entire quotient of $4m^2$ by d, d', d'', d''' ,

$$x = 2m + q, 2m + q', 2m + q'', 2m + q''',$$

The number of solutions at first appears equal to the number of the divisors of $4m^2$; but as the equation is symmetrical* in x and y ,

* We call a symmetrical function of two or more quantities any expression which contains these quantities combined in the same manner; that is, such that when we change these quantities the one into the other, the expression does not change except in the order of the terms.

the values of x , deduced from the equation first resolved with reference to y , will be the same as those of y taken in the inverse order. Thus, the whole number of *distinct solutions* is really equal to only *half the number of divisors*, if this number is even, and to *half the number plus unity* if it is odd.

As a second application, let

$$m = 3, \text{ whence } x = 6 + \frac{36}{y-6};$$

seeking the divisors of 36, we have

$$y - 6 = 1, 2, 3, 4, 6, 9, 12, 18, 36;$$

$$\text{whence } y = 7, 8, 9, 10, 12, 15, 18, 24, 42.$$

$$\frac{36}{y-6} = 36, 18, 12, 9, 6, 4, 3, 2, 1.$$

$$\text{Then } x = 42, 24, 18, 15, 12, 10, 9, 8, 7;$$

which gives *five distinct solutions*.

144. Question 2. *Having given the side a of a square, to find in whole numbers the sides of a rectangle whose perimeter shall be to that of the square in the same ratio as their surfaces.*

Let x and y be the sides of the rectangle; $2(x+y)$ and xy represent its perimeter and its surface. Moreover, $4a$ and a^2 express the perimeter and the surface of the square; so that we have the equation

$$\frac{xy}{a^2} = \frac{2(x+y)}{4a},$$

$$\text{or simplifying } 2xy = a(x+y).$$

This may present two cases, when a is an even and when it is an odd number.

(1.) If a is an even number and equal to $2a'$, it becomes, by suppressing the factor 2,

$$xy = a'(x+y), \text{ whence } x = a' + \frac{a'^2}{y-a'},$$

and the question returns to the preceding.

(2.) If it is an odd number, we have

$$x = \frac{ay}{2y-a}, \text{ or } x = \frac{a}{2} + \frac{a^2}{2(2y-a)}.$$

In order that this expression of x may be entire, it is evidently only necessary that $2y-a$ should be a divisor of a^2 .

By designating by d, d', d'' , these divisors, and by q, q', q'' , the quotients of the division of a^2 by d, d', d'' , (the quantities d, d', d'' ,

q, q', q'' , are necessarily odd numbers, since by hypothesis a is an odd number), we shall have

$$2y - a = d, \quad d', \quad d'', \text{ odd numbers;}$$

whence

$$y = \frac{a+d}{2}, \quad \frac{a+d'}{2}, \quad \frac{a+d''}{2}, \text{ entire expressions.}$$

$$\frac{a^2}{2y-a} = q, \quad q', \quad q'', \text{ odd numbers,}$$

and consequently,

$$x = \frac{a+q}{2}, \quad \frac{a+q'}{2}, \quad \frac{a+q''}{2}, \text{ entire expressions.}$$

In the first place, let $a = 20$; the equation is

$$2xy = 20(x+y),$$

or, dividing by 2, and resolving with reference to x ,

$$x = 10 + \frac{100}{y-10};$$

seeking the divisors of 100, we find

$$y - 10 = 1, 2, 4, 5, 10, 20, 25, 50, 100;$$

whence

$$y = 11, 12, 14, 15, 20, 30, 35, 60, 110;$$

$$\frac{100}{y-10} = 100, 50, 25, 20, 10, 5, 4, 2, 1;$$

then

$$x = 110, 60, 35, 30, 20, 15, 14, 12, 11,$$

which gives *five different solutions*.

In the second place, let $a = 15$; we have the equation

$$2xy = 15(x+y);$$

whence we deduce

$$x = \frac{15y}{2y-15}, \text{ or } x = \frac{15}{2} + \frac{225}{2(2y-15)}.$$

Seeking the divisors of 225, we obtain

$$2y - 15 = 1, 3, 5, 9, 15, 25, 45, 75, 225;$$

whence

$$y = 8, 9, 10, 12, 15, 20, 30, 45, 120;$$

$$\frac{225}{2y-15} = 225, 75, 45, 25, 15, 9, 5, 3, 1;$$

then

$$x = 120, 45, 30, 20, 15, 12, 10, 9, 8;$$

in all, *five different solutions.*

145. Question 3. *Having given the side a of a cube, in a whole number, it is required to find, also in whole numbers, the side of the base and the altitude of a rectangular parallelopiped, with a square base, such that their solidities may be to each other as their surfaces.*

Let x be the side of the base, and y the altitude of this parallelopiped, $x^2 y$ and $2 x^2 + 4 x y$ will represent the solidity and the surface of this solid; moreover, a^3 and $6 a^2$ are the expressions for the solidity and the surface of the given cube; we have then the equation .

$$\frac{x^2 y}{2 x^2 + 4 x y} = \frac{a^3}{6 a^2},$$

or, by freeing from denominators and reducing,

$$3 x y = a x + 2 a y.$$

From this equation we deduce

$$x = \frac{2 a y}{3 y - a} = \frac{2 a}{3} + \frac{2 a^2}{3(3 y - a)},$$

or,
$$3 x = 2 a + \frac{2 a^2}{3 y - a}.$$

This being premised, let us designate by d, d', d'' , all the divisors of $2 a^2$, and let us put

$$3 y - a = d, d', d'';$$

there hence results

$$y = \frac{a + d}{3}, \frac{a + d'}{3}, \frac{a + d''}{3}.$$

Moreover, let q, q', q'' , be the quotients of $2 a^2$ by d, d', d'' , we have

$$\frac{2 a^2}{3 y - a} = q, q', q'';$$

whence

$$3 x = 2 a + q, 2 a + q', 2 a + q''.$$

Then

$$x = \frac{2 a + q}{3}, \frac{2 a + q'}{3}, \frac{2 a + q''}{3}.$$

By taking account of entire expressions only in the two series of values of x and y , we shall obtain the sets of values in whole numbers which are capable of verifying the equation.

For example, let $a = 8$, the equation becomes

$$3x = 16 + \frac{128}{3y-8},$$

seeking the divisors of 128, and making them equal to $3y - 8$, we have

$$3y - 8 = 1, 2, 4, 8, 16, 32, 64, 128,$$

whence

$$y = 3, 4, 8, 24, \\ \frac{128}{3y-8} = 128, 32, 8, 2;$$

then $3x = 16 + 128, 16 + 32, 16 + 8, 16 + 2$,
and consequently,

$$x = 48, 16, 8, 6.$$

Thus, this particular equation admits of the sets

$$x = 48, 16, 8, 6,$$

$$y = 3, 4, 8, 24.$$

We propose for a new exercise the equations

$$(1.) \quad xy = 2x + 2y + 20 \quad \begin{cases} x = 26, 14, 10, 8, \\ y = 3, 4, 5, 6. \end{cases}$$

$$(2.) \quad 8xy = 6x + 5y + 12; \text{ the only set, } y = 6, x = 1.$$

$$(3.) \quad 4xy = 3x + 2y - 12 \quad \begin{cases} x = 0, 4, \\ y = 6, 0. \end{cases}$$

$$(4.) \quad \text{The general equation } mx y = ax + by + c.$$

Question 4. *It is required to find, in whole numbers, the rectangular parallelepipeds with square bases, such that their solidities may contain five times as many cubic feet as their surfaces contain square feet.*

Side of the base, $x = 220, 120, 70, 60, 45, 40, 30, 28, 25, 24, 22, 21$,
altitude $y = 11, 12, 14, 15, 18, 20, 30, 35, 50, 60, 110, 210$.

Twelve solutions.

Of the Formation of Powers and the Extraction of Roots of any Degree whatever.

Introduction.—As the resolution of equations of the second degree supposes the process for the extraction of the square root already known, so, the resolution of equations of the third and

fourth degrees requires that we should know how to extract the roots of the third and fourth degrees of a quantity, either numerical or algebraic. (See article 2, for the definitions of the words, *power* and *root*.)

The raising of powers, and the extraction of the roots of every degree, and the calculation of radicals, form the principal subject of this section, which, with the first and a part of the third, comprehend together the operations which we may have to perform upon numbers expressed algebraically.

Although any power of a number may be obtained by means of the rules of multiplication, either arithmetically or algebraically, nevertheless this power is subject to a *law of composition* which must be known if we would *return from the power to its root*. Now as the law of the composition of the square of any quantity, either numerical or algebraical, is founded upon the expression of the square of a binomial, so the law relative to a power of any degree whatever is deduced from the expression of the power of a binomial of the same degree. It is then by the determination of *the developement of any power of a binomial*, that we ought to commence this new theory.

I. Binomial Theorem of Newton and the Consequences which are derived from it.

146. If we multiply the binomial $x + a$ several times by itself, we arrive at the following results.

$$(x + a)^1 = x + a,$$

$$(x + a)^2 = x^2 + 2ax + a^2,$$

$$(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3,$$

$$(x + a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4,$$

$$(x + a)^5 = x^5 + 5ax^4 + 10a^2x^3 + 10a^3x^2 + 5a^4x + a^5.$$

By looking at these different developements, we easily recognise the law according to which they proceed, as it regards the exponents of x and a ; it is not the same with respect to the coefficients. But Newton arrived at a law, by means of which, the degree of the power being given, we are enabled to form this power of a binomial without being obliged to go through the inferior powers. He has left no traces of the reasoning by which he was led to this conclusion, but the truth of the law has since been established in the most

rigorous manner. Of all known demonstrations, the most elementary is that which is founded upon the *Theory of Combinations*. Still, as it is somewhat complicated, in order to simplify the explanation, we shall begin by resolving some problems relative to combinations, from which it will be easy to deduce the *binomial formula*, or the development of any power whatever of a binomial.

147. *Preliminary considerations.* We know already (Lacroix's Arithmetic, article 27), that the product of a number n of factors a, b, c, d , is not changed in whatever order we perform their multiplication. Now it is proposed to determine the *whole number* of ways in which these letters are capable of being placed by the side of each other. The results answering to the changes thus made among these letters, are called *permutations*.

For instance, two letters a and b , which give only one product ab , furnish two permutations, ab and ba .

In like manner, the three letters a, b, c , give only one product, abc , but furnish six permutations,

$$abc, acb, cab, bac, bca, cba,$$

Let there be now a number m of letters $a, b, c, d, \dots$; if we place these one after the other, 2 and 2, 3 and 3, 4 and 4, in every possible order, in such a manner, that in each result the number of letters may be less than the number of letters given, and the *whole number* of the results thus obtained is sought. These results are what are called *arrangements*.

Thus, $ab, ac, ad, ba, bc, bd, ca, cb, cd, \dots$, are the arrangements of m letters, taken 2 and 2.

Also, $abc, abd, bac, bad, acb, acd, \dots$, are the arrangements, taken 3 and 3.

Finally, when we have thus disposed the letters by the side of each other, 2 and 2, 3 and 3, 4 and 4, we may require that no two results thus formed, shall be composed of the same letters, that is, that they shall differ from each other, at least by one letter, and we may seek the whole number of results which can be obtained in this way. In this case the results take the name of *combinations*.

Thus, $ab, ac, bc, ad, bd, \dots$, are combinations of 2 and 2, in which any two of the results differ at least in one of the letters.

Also $abc, abd, acd, bcd, \dots$, are combinations of 3 and 3.

There is then an essential distinction in the signification of the words *permutation*, *arrangement*, and *combination*.

We give the name **PERMUTATION** to the results which we obtain by disposing one after another, and in all possible orders, a determinate number of letters, in such a manner that all the letters enter into each result, and that each letter is found in this result but once.

The name **ARRANGEMENTS** is applied to the results which we obtain by placing one after another, and in all possible orders, a number m of letters, taken 2 and 2, 3 and 3, 4 and 4, . . . n and n , m being $> n$, that is, the number of letters in each result being less than the whole number of letters made use of. If, however, we suppose $m = n$, the arrangements n and n become simple permutations.

Finally, we call **COMBINATIONS**, the arrangements in which any two differ from each other at least in one of the letters which compose them.

It is important that the learner should be well initiated in these definitions in order to understand clearly the resolution of the following problems.

148. Problem 1. *To determine the whole number of permutations of which n letters are susceptible.*

In the first place, two letters, a and b , evidently give two permutations, ab and ba . Thus, the number of permutations of two letters is 2, or 1×2 .

Let there be three letters, a, b, c . Take any one of these letters, c , for example, and write it on the right of the two arrangements ab , and ba , which the two letters give, and there results two permutations of three letters, abc , bac . But as we can take separately either of these letters, it follows that the whole number of permutations of three letters is equal to 2×3 , or $1 \times 2 \times 3$.

In general, let there be a number n of letters, a, b, c, d , and let us suppose the whole number of permutations of $n - 1$ letters already known, and let this number be designated by Q .

Let us consider separately one of the n letters, and write this letter on the right of each of the Q permutations which the $n - 1$ other letters give; there will result Q permutations of n letters, terminated by the letter which we at first insulated. But as we can thus take separately each one of the n letters, it follows that the whole number of permutations of n letters is equal to $Q \times n$.

Let $n = 2$, Q will then designate the number of permutations that a single letter can give; then $Q = 1$, and we have in this particular case, $Q \times n = 1 \times 2$.

Let $n = 3$, Q will then express the number of permutations of 3 — 1, or of 2 letters, and is equal to 1×2 . So that $Q \times n$ is reduced to $1 \times 2 \times 3$.

Farther, let $n = 4$, Q will designate in this case the number of permutations of 3 letters, and is equal to $1 \times 2 \times 3$. Then $Q \times n$ becomes $1 \times 2 \times 3 \times 4$.

N. B. We see then, that the formula $Q \times n$ includes all the particular cases of the proposed problem. By observing the above reasoning we can immediately determine the general case, and thence deduce at the same time all the particular cases.

149. Problem 2. *A number m of letters, a, b, c, d, being given to determinē the whole number of arrangements which can be formed with these m letters, taken n at a time, m being supposed greater than n.*

To resolve at once this general question, let us suppose already known the whole number of the *arrangements* which can be formed with the m letters, $n - 1$ at a time, and let us designate this number by P .

Let us consider any one of these arrangements and set down at the right hand each of the letters which do not enter into this arrangement, and of which the number is necessarily

$$m - (n - 1) \text{ or } m - n + 1;$$

it is evident, that we shall thus form a number $m - n + 1$ of arrangements of n letters, each differing from the others by the last letter.

Let us consider a new arrangement of $n - 1$ letters, and write at the right hand the $m - n + 1$ letters which do not enter into this arrangement, we shall still obtain a number $m - n + 1$ of arrangements of n letters, differing from each other, and from the preceding arrangements at least by the disposition of one of the $n - 1$ first letters. Since we may consider separately each of the P arrangements of the letters, taken $n - 1$ at a time, and write successively at the right hand the $m - n + 1$ other letters, it follows that the whole number of arrangements of m letters, taken n at a time, is expressed by

$$P(m - n + 1).$$

To find now, as a particular case, the whole number of the arrangements of m letters, taken 2 and 2, 3 and 3, 4 and 4.

Take $n = 2$, whence we have $m - n + 1 = m - 1$; P expresses, in this case, the whole number of arrangements taken 2 — 1 at a time, or taken one at a time, and is consequently equal to m ; then the formula becomes $m(m - 1)$.

Let $n = 3$, whence $m - n + 1 = m - 2$; P then expresses the number of arrangements, taken 2 and 2, and is equal to $m(m - 1)$; then the formula becomes $m(m - 1)(m - 2)$.

Further, let $n = 4$, whence $m - n + 1 = m - 3$; P expresses the number of arrangements, taken 3 and 3, or is equal to

$$m(m - 1)(m - 2);$$

then the formula becomes $m(m - 1)(m - 2)(m - 3)$; and so on.

N. B. According to the manner in which particular cases have been deduced from the general formula $P(m - n + 1)$, we may conclude that this formula developed, becomes

$$m(m - 1)(m - 2)(m - 3) \dots (m - n + 1);$$

that is, *it is composed of the product of the n consecutive and decreasing numbers, which are comprehended from m to*

$$m - (n - 1) \text{ or } m - n + 1, \text{ inclusive.}$$

This being laid down, it is easy to deduce from this formula developed, the formula of the preceding article, that is, the value of $Q \times n$ also developed.

We have seen (147) that the *arrangements* become permutations when we suppose the number of letters which enters into each arrangement equal to the whole number of letters considered.

Thus, to pass from the whole number of arrangements of m letters, taken n at a time, to the number of permutations of n letters, we have only to make, in the above developement, $m = n$; which gives

$$n(n - 1)(n - 2)(n - 3) \dots 1.$$

Reversing the order of the factors and observing that the last factor being 1, the last but one is 2, the next 3, we have

$$1 \cdot 2 \cdot 3 \cdot 4 \dots (n - 2)(n - 1)n$$

for the developement of $Q \times n$.

This is only the natural order of the numbers, comprehended between 1 and n inclusive.

150. Problem 3. *To determine the whole number of different combinations which can be formed with m letters taken n at a time.*

Let us designate by X the whole number of *arrangements* which can be formed with m letters, taken n at a time by Y , the number of permutations of which n letters are susceptible, and by Z the whole number of *different combinations*, taken n at a time, the number which it is required to determine.

It is evident, that in order to obtain all the possible arrangements of m letters, taken n at a time, it is sufficient to give to the n letters of each of the Z combinations, all the permutations of which these letters are susceptible.

Now a single combination of n letters gives by hypothesis Y permutations; then Z combinations of n letters must give $Y \times Z$ arrangements, taken n at a time; and as we have already designated by X the whole number of the arrangements, it follows that the three quantities X , Y , Z , are connected by the relation

$$X = Y \times Z,$$

whence we deduce $Z = \frac{X}{Y}$.

But we have found (149)

$$X = P(m - n + 1),$$

and (148)

$$Y = Q \times n.$$

Then, finally,

$$Z = \frac{P(m - n + 1)}{Q \times n} = \frac{P}{Q} \times \frac{m - n + 1}{n}.$$

Since P expresses the whole number of arrangements, taken $n - 1$ at a time, and Q expresses the whole number of permutations of $n - 1$ letters, it follows that $\frac{P}{Q}$ expresses the number of different combinations of m letters, taken $n - 1$ at a time.

Accordingly, if we wish to find as particular cases, the number of combinations 2 and 2, 3 and 3, 4 and 4 . . . ;

let us make $n = 2$, in which case $\frac{P}{Q}$ expresses the number of combinations, taken 2 — 1 at a time, or 1 at a time, and is equal to m ; the formula above becomes

$$m \times \frac{m-1}{2} \text{ or } \frac{m(m-1)}{1 \cdot 2}.$$

Let us make $n = 3$, in which case $\frac{P}{Q}$ expresses the number of

combinations, 2 and 2, or is equal to $\frac{m(m-1)}{2}$; the formula becomes

$$\frac{m(m-1)m(m-2)}{1 \cdot 2 \cdot 3}$$

We may find, in the same manner,

$$\frac{m(m-1)(m-2)(m-3)}{1 \cdot 2 \cdot 3 \cdot 4}$$

for the number of combinations, 4 and 4, &c., and in general, for the number of the combinations, n and n , we have

$$\frac{m(m-1)(m-2)(m-3)\dots(m-n+1)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \dots n-1 \cdot n}$$

that is, the expression

$$\frac{P(m-n+1)}{Q \times n} \text{ developed.}$$

151. *Demonstration of the binomial formula.* In order to discover more easily the law of the developement of the m^{th} power of the binomial $x + a$, we shall begin by observing the law of the product of several binomials

$$x + a, x + b, x + c, x + d, \dots$$

having a first term common, and whose second terms are different.

	$x + a$			
	$x + b$			
1st product	$\begin{array}{r} x^2 + a \\ + b \end{array}$	$x + ab$		
	$x + c$			
2d product	$\begin{array}{r} x^3 + a \\ + b \\ + c \end{array}$	$\begin{array}{r} x^2 + ab \\ + ac \\ + bc \end{array}$	$x + abc$	
	$x + d$			
3d product	$\begin{array}{r} x^4 + a \\ + b \\ + c \\ + d \end{array}$	$\begin{array}{r} x^3 + ab \\ + ac \\ + ad \\ + bc \\ + bd \\ + cd \end{array}$	$\begin{array}{r} x^2 + abc \\ + abd \\ + acd \\ + bcd \end{array}$	$x + abcd$

These multiplications being performed according to the common rules of algebraic multiplication, we discover by the three preceding products the following law;

(1.) With respect to the exponents, the exponent of x is equal to the number of the binomial factors. This exponent goes on decreasing by unity in each term till the last, where it is equal to zero.

(2.) With respect to the coefficients of the different powers of x , the coefficient of the first term is unity; the coefficient of the second term is equal to the sum of the second terms of the binomials; the coefficient of the third term is equal to the sum of the different products of these same second terms multiplied two and two; the coefficient of the fourth term is equal to the sum of the different products, three and three. Following this *analogy*, we say that the coefficient of a term, having n terms before it, is equal to the sum of the different products of the second terms of the binomials, taken n at a time. Finally, the last term is equal to the product of the second terms of the binomials.

To be convinced that this law is general, let us suppose it already verified for the product of a number m of binomials, and see if it applies when we introduce a new factor into the product.

Let then

$x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + \dots Mx^{m-n+1} + Nx^{m-n} + \dots + U$,
the product of m binomial factors (Nx^{m-n} represents a term which has n terms before it, and Mx^{m-n+1} that which immediately precedes it.)

Moreover, let $x + K$ be the new factor introduced; we have for the product, arranged with reference to x ,

$$\begin{array}{ccccccc} x^{m+1} + A & | & x^m + B & | & x^{m-1} + C & | & x^{m-2} + \dots + N & | & x^{m-n+1} + \dots \\ + K & | & + AK & | & + BK & | & + MK & | & + UK. \end{array}$$

Already the law of the exponents is evidently the same.

As to the coefficients, (1.) that of the first term is unity, (2.) $A + K$, or the coefficient of x^m , is also the sum of the second terms of the $m + 1$ binomials.

(3.) B is, by hypothesis, equal to the sum of the different products of the second terms of the m first binomials, taken 2 and 2; AK expresses the sum of the products of each of the second terms of the m first binomials, multiplied by the new second term K ; then $B + AK$ is also the sum of the different products of the second terms of the $m + 1$ binomials, taken two and two.

In general, since N expresses the sum of the products of the second terms of the m first binomials, taken n at a time, and MK

represents the sum of the products of these second terms, taken $n-1$ at a time, multiplied by the new second term K , it follows that $N + MK$, or the coefficient which in the polynomial of the degree $m+1$, having n terms before it, is equal to the sum of the products of the second terms of the $m+1$ binomials, taken n at a time. The last term, $U \times K$, is moreover equal to the product of the $m+1$ second terms.

Thus the law of composition, supposed true for the product of a number m of binomials, is so likewise for a number $m+1$; then it is general.

Let us suppose that in the product of m binomial factors

$$x + a, x + b, x + c, x + d \dots,$$

we make

$$a = b = c = d;$$

the expression indicating this product,

$$(x+a)(x+b)(x+c)(x+d) \dots \text{ is changed into } (x+a)^m.$$

As to its developements, the coefficient being

$$a + b + c + d + \dots, ab + ac + ad + \dots, abc + abd + acd + \dots,$$

(1.) the coefficient of x^{m-1} , or $a + b + c + d + \dots$, becomes

$$a + a + a + a + \dots;$$

that is, a taken as many times as there are letters $a, b, c \dots$, and is reduced consequently to ma . (2.) The coefficient of x^{m-2} , or $ab + ac + \dots$ is reduced to $a^2 + a^2 + a^2 \dots$, or rather to as many times a^2 as we can form different combinations of the m letters, multiplied 2 and 2, or (150) in short to

$$m \cdot \frac{m-1}{2} a^2.$$

(3.) The coefficient of x^{m-3} is reduced to the product of a^3 , multiplied by the number of different combinations of m letters taken 3 and 3, or rather to

$$m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot a^3;$$

and so on.

In general, if we designate by Nx^{m-n} the term which has a number n before it, the coefficient N , which in the hypothesis where the second terms of the binomials are different, is equal to the sum of their products, taken n at a time, is reduced, when we suppose them all equal, to a^n multiplied by the number of different combina-

tions given by m letters, taken n at a time. Thus (150),

$$N = \frac{P(m-n+1)}{Q \times n} a^n.$$

Finally, then, we have the formula

$$(x+a)^m = x^m + m a x^{m-1} + m \frac{m-1}{2} a^2 x^{m-2} \\ + m \frac{m-1}{2} \cdot \frac{m-2}{3} a^3 x^{m-3} \dots + \frac{P(m-n+1)}{Q \cdot n} a^n x^{m-n} \dots + a^m.$$

152. If we look at the different terms of this developement, we shall discover a simple law according to which a coefficient of whatever place is formed by means of the preceding coefficient.

The coefficient of a term of any place is formed by multiplying the coefficient of the preceding term by the exponent of x in this term, and dividing the product by the number of terms which precede that which is considered.

For, let us take the general term

$$\frac{P(m-n+1)}{Q \cdot n} a^n x^{m-n},$$

(we call it the *general term*, because in taking successively $n = 2, 3, 4, \dots$, we can deduce from it all the others); the term which precedes it by one place, is evidently

$$\frac{P}{Q} a^{n-1} x^{m-n+1},$$

since (150) $\frac{P}{Q}$ expresses the number of combinations, taken $n-1$ at a time.

Now we see that the coefficient

$$\frac{P(m-n+1)}{Q \cdot n}$$

is equal to the coefficient $\frac{P}{Q}$ which precedes it, multiplied by $m-n+1$, the exponent of x in this term, and divided by n , the number of the terms which precede that which we are considering. It is in this law, for which we are indebted to Newton, that the *binomial formula* principally consists. It serves to develope a particular power even without our being obliged to have recourse to the general formula.

For example, let it be proposed to develope $(x+a)^6$. We shall find, according to this law,

$$(x+a)^6 = x^6 + 6ax^5 + 15a^2x^4 + 20a^3x^3 + 15a^4x^2 + 6a^5x + a^6.$$

After having formed the two first terms, in which there is no difficulty, according to the terms of the general formula

$$x^m + m a x^{m-1} +,$$

we multiply 6, the coefficient of the second term, by 5, the exponent of x in that term, then we divide the product by 2, which gives 15 for the coefficient of the third term. In order to obtain that of the fourth, we multiply 15 by 4, the exponent of x in the third term, and divide the product by 3, the number of terms which precede the fourth, which gives 20; and so on for all the other terms.

We should find, in a similar manner,

$$(x + a)^{10} = x^{10} + 10ax^9 + 45a^2x^8 + 120a^3x^7 + 210a^4x^6 \\ + 252a^5x^5 + 210a^6x^4 + 120a^7x^3 + 45a^8x^2 + 10a^9x + a^{10}.$$

We shall resume this subject in the sequel with reference to the powers of algebraic expressions.

Consequences of the Binomial Formula and of the Theory of Combinations.

153. *Consequence First.*—The expression $(x + a)^m$ being composed in the same manner of a and x , it ought to be so likewise in its developement; then if this developement includes a term of the form $Ka^n x^{m-n}$, it ought to have another equal to $Kx^n a^{m-n}$, or $Ka^{m-n} x^n$. These two terms are evidently at an equal distance from the two extremes in the developement; for the number of the terms which precedes any term whatever, being marked by the exponent of a in this term, it follows that the term $Ka^n x^{m-n}$ has n terms before it, that the term $Ka^{m-n} x^n$ has a number $m - n$ before and consequently n after it, since the whole number of terms is $(m + 1)$.

Thus, in the developement of every power of a binomial the coefficients at equal distances from the two extremes are equal.

N. B. In the terms $Ka^n x^{m-n}$, $Ka^{m-n} x^n$, the two coefficients express the numbers of different combinations which can be formed with m quantities taken n at a time, and $m - n$ at a time; thus we may still conclude that the number of different combinations of m

quantities taken n at a time, is equal to the number of combinations of these same quantities taken $m - n$ at a time.

For example, twelve quantities combined, 5 and 5, give the same number of combinations as these twelve quantities combined, 12 — 5 at a time, or 7 and 7. Five quantities combined, 2 and 2, give the same number of combinations as five quantities combined, 5 — 2 at a time, or 3 and 3.

154. *Consequence Second.* If in the general formula

$$(x + a)^m = x^m + m a x^{m-1} + m \frac{m-1}{2} a^2 x^{m-2} + \&c.,$$

we suppose $x = 1, a = 1$, it becomes

$$(1 + 1)^m \text{ or } 2^m = 1 + m + m \frac{m-1}{2} + m \frac{m-1}{2} \cdot \frac{m-2}{3} + \&c.;$$

that is, the sum of the coefficients of the different terms of the binomial formula is equal to that power of 2 whose exponent is m .

Thus, in the particular formula

$$(x + a)^5 = x^5 = 5 a x^4 + 10 a^2 x^3 + 10 a^3 x^2 + 5 a^4 x + a^5,$$

the sum $1 + 5 + 10 + 10 + 5 + 1$ of the coefficients equals 2^5 , or 32. In the tenth power, developed article 152, the sum of the coefficients is equal to 2^{10} , or 1024.

155. *Consequence Third.* When we have a series of numbers decreasing by unity from one term to the other, of which the first is m , and the last $m - p$ (m and p are whole numbers), if we make a single product of all these numbers, this product is divisible by the product of all the natural numbers from 1 to $p + 1$. That is, we have

$$\frac{m(m-1)(m-2)(m-3) \dots (m-p)}{1 \cdot 2 \cdot 3 \cdot 4 \dots p+1}$$

equal to a whole number. For, it follows from what has been said (150), that this expression represents the number of different combinations which may be formed of m letters, taken $p + 1$ at a time. Now this number of combinations must be, by its nature, a whole number; then the above expression is necessarily a whole number.

We recommend to the learner to seek a demonstration of this property independently of the Theory of Combinations and of the binominal theorem, apprising him that the question which is suffi-

is easily easy with respect to the first expressions

$$\frac{m(m-1)}{1 \cdot 2}, \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3},$$

becomes more difficult in the general case.

II. Of the Extraction of the Roots of Particular Numbers.

Although we have explained in our Arithmetic the principles of the extraction of the *cube root*, it seems necessary to introduce here this process; *first*, because next to the extraction of the square root it is an operation for which we have the most frequent occasion; *secondly*, because the same reasonings may be applied to the extraction of the 4th, 5th, and in general of the n^{th} root.

156. We call the *cube* or *third power* of a number the product of that number multiplied twice by itself, and the *third* or *cube root* the number which, raised to the cube, will produce the proposed number.

The ten first numbers being

1, 2, 3, 4, 5, 6, 7, 8, 9, 10,

we have for their cubes

1, 8, 27, 64, 125, 216, 343, 512, 729, 1000.

Reciprocally, the numbers in the second line have for their cube roots the numbers of the first line.

We discover, by inspecting these lines, that among numbers of one, two, and three figures, there are only nine which are *perfect cubes*; each of the others has for its cube root a whole number plus a fraction which *cannot be expressed exactly by means of unity*.

For, let us suppose for a moment that $\frac{a}{b}$, an irreducible fractional number, is the root of a whole number N ; it would follow that the cube of $\frac{a}{b}$ or $\frac{a^3}{b^3}$, would be equal to N . Now that is impossible, for a and b being prime to each other, it is the same with a^3 and b^3 ; then $\frac{a^3}{b^3}$ cannot be equal to a whole number.

157. The difference between *two consecutive perfect cubes* is so much the greater as the two roots are greater, and this difference may be easily estimated.

For, let a and $a + 1$ be two consecutive whole numbers ; we have (152),

$$(a + 1)^3 = a^3 + 3a^2 + 3a + 1 ;$$

whence $(a + 1)^3 - a^3 = 3a^2 + 3a + 1 ;$

that is, *the difference of two consecutive cubes is equal to three times the square of the smallest root, plus three times this same root, plus 1.*

Thus the difference between the cube of 90 and the cube of 89 is equal to

$$3(89)^2 + 3 \times 89 + 1 = 24031.$$

158. Let us now seek a process for extracting the cube root of a whole number.

First, if the number has only three figures at most, *its root is immediately obtained by reference to the cubes of the nine first numbers.* Thus the cube root of 125 is 5 ; the cube root of 72 is 4, plus a fraction, or is 4 within unity, the cube root of 841 is 9, within unity, since 841 falls between 729 or the cube of 9, and 1000 or the cube of 10. Let us then consider a number of more than three figures.

For example, let 103823 be the number proposed.

103. 823	47	
64	48	
398. 23	48	47
	48	47
	384	329
	192	188
	2304	2209
	48	47
	18432	15463
	9216	8836
	110592	103823

This number being comprehended between 1000, which is the cube of 10, and 1000000, which is the cube of 100, its root is necessarily composed of two figures, or of tens and units. Let us designate by a the tens and by b the units, we have (46)

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Whence it follows that the cube of a number, composed of tens and units, contains *the cube of the tens, the triple product of the*
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square of the tens by the units, the triple product of the square of units by the tens, plus the cube of the units.

This being laid down, the cube of the tens giving at least thousands, the three last significant figures on the right do not enter into it, and it is in the portion 103 (which we separate from the three last figures by a point) that we find the cube of the tens. Now the root of the greatest cube contained in 103 being 4, the root of 64, 4 is the figure of the tens of the root sought; for 103823 is evidently comprehended between $(40)^3$ or 64000, and $(50)^3$ or 125000; then the root required is composed of 4 tens, plus a certain number of units less than ten.

The figure of the tens being obtained, let us subtract the cube 64 from 103; there remains 39, which, with the remaining portion 823, gives 39823; and this result contains still *three times the square of the tens multiplied by the units*, plus the two other parts above named. Now the square of a number of tens containing no figure inferior to hundreds, it follows, that this triple product can be found only in the part 398, on the left of the two last figures 23, (which we separate for that reason by a point). On the other hand, we can form the triple square of the 4 tens, which gives 48; then if we divide 398 by 48, the quotient 8 is the figure of the units of the root, or a figure too great, because the 398 hundreds are composed of the triple product of the square of the tens multiplied by the units, and the two remaining parts.

In order to ascertain if this figure 8 is not too great, we can, as for the square root, form, by the aid of this figure 8 and of the figure 4 of the tens, the three parts which are contained in 39823; but it is much more simple to raise 48 to the cube (as we did in the foregoing table). Now, we find for this cube 110592, a number greater than 103823; so that the figure 8 is too large. In forming the cube of 47, we obtain 103823; thus the proposed number is a perfect cube, and has for its cube root 47.

N. B. We cannot immediately find the figure of the units; for the cube of the units (156) giving tens and even hundreds, these tens and hundreds are involved with those which proceed from the other parts of the cube.

Again, let it be required to extract the cube root of 47954.

47. 954	36
27	27
209	36
	36
47954	216
46656	108
1298	1296
	36
	7776
	3888
	46656

The number 47954 being less than 1000000, its root has only two figures, that is, only tens and units. The cube of the tens is found in the 47 *thousands*, and we can prove as before, that 3, the root of the greatest cube contained in 47, expresses the tens of the root. Let us subtract the cube of 3 or 27 from 47, there remains 20; bring down to this remainder the single figure 9 of the portion 954; the number 209 hundreds is composed of the triple product of the square of the tens multiplied by the units, and figures arising from the two other portions. Then if we form the triple square of the tens 3, which gives 27, and divide 209 by 27, the quotient 7 is the figure of the units of the root, or a figure too great. By raising 37 to the cube, we find 50653, a number greater than 47954; if we form then the cube of 36, we obtain 46656, a number which subtracted from 47954 (as is seen in the foregoing table), gives for a remainder 1298. Thus, the proposed number is not a perfect cube; but its root, *within unity*, is 36. For the difference between the proposed number and the cube of 36, is, as we have just seen, 1298, a number smaller than

$$3 \times (36)^2 + 3 \times 36 + 1,$$

since we have obtained, in the course of the verification, 3888 for three times the square of 36.

159. Let it be now proposed to extract the cube root of a number of more than 6 figures, of 43725658, for example.

43 . 725 . 658	352	
27	27 3675	
167	35	352
	35	352
43 725	175	704
42 875	105	1760
8506	1225	1056
	35	123904
43725658	6125	352
43614208	3675	247808
rem. 111450	42875	619520
		371712
		43614208

Whatever may be the root sought, it has necessarily more than one figure, and we may consider it as composed of units and tens only (as the tens may be expressed by one or more figures).

Now the cube of the tens gives at least thousands; thus, it is necessarily found in the portion at the left of the three last figures 658. If now we extract the root of the greatest cube contained in the portion on the left, 43725, considered in its absolute value, we shall have the whole number of the tens of the root; for, let a be the root of 43725, within unity, that is, let a^3 and $(a + 1)^3$ include 43725, then also 43725000 is comprehended between

$$a^3 \times 1000 \text{ and } (a + 1)^3 \times 1000;$$

and as these two last numbers differ from each other by more than 1000, it follows that the proposed number itself, or 43725658, is comprehended between

$$a^3 \times 1000 \text{ and } (a + 1)^3 \times 1000;$$

thus the root sought is comprehended between those of

$$a^3 \times 1000 \text{ and } (a + 1)^3 \times 1000;$$

that is, between

$$a \times 10 \text{ and } (a + 1) \times 10.$$

Then in short, it is composed of a tens and a certain number of units less than ten.

The question is then reduced to extracting the cube root of 43725; but this new number having more than three figures, its

root has more than one, that is, it contains tens and units. In order to obtain the tens, we must separate the three last figures 725, and extract the root of the greatest cube contained in 43. (It is sufficiently evident what would be necessary, if this new number had more than three figures).

The greatest cube contained in 43 is 27, of which the root is 3, and this figure expresses then the tens of the root of 43725 (or the figure of the hundreds of the whole root). Subtracting the cube of 3, or 27, from 43, we obtain 16 for the remainder, to which we must bring down the first figure 7 of the portion 725, which gives 167.

Forming the triple square of the tens 3, we find 27; and if we divide 167 by 27, the quotient 6 is the figure of the units of the root of 43725, or a figure too large; it is easy to foresee that this figure is indeed too great; hence we must make trial of 5, and for that purpose let us raise 35 to the cube; the result is 42875, a number which, subtracted from 43725, gives for a remainder 850, a number evidently smaller than

$$3 \times (35)^2 + 3 \times 35 + 1.$$

So that 35 is the root of the greatest cube contained in 43725; it is then the number of the tens of the root sought.

In order to obtain the units, we bring down to the remainder 850, the first figure 6 of the last portion 658, which gives 8506; we form moreover the triple square of the tens 35, (this is easily done, since in the preceding verification, we have already formed the square of 35); then we divide 8506 by this triple square, 3675; the quotient is 2; let us make trial of it by raising 352 to the cube, which gives 43614208, a result smaller than the proposed number; by subtracting it from the latter, we obtain for a remainder 111450. Thus, 352 is the cube root of 43725658, within unity.

General Rule. In order to extract the cube root of a whole number, separate the number into portions of three figures each, proceeding from the right, till you arrive at a portion of one, two, or three figures at most (the number of portions is equal to the number of figures of the root); extract the root of the greatest cube contained in the first portion on the left, and subtract this cube from the first portion; bring down to the remainder the first figure of the second portion and divide the number thus formed by the triple

square of the figure of the root already found; set down the quotient on the right of this figure, and raise the two figures to the cube; if this cube is greater than the two first portions, diminish the quotient by one or more units, till you obtain a cube which may be subtracted from the two first portions; the subtraction being made, to the remainder bring down the first figure of the third portion, then divide the number thus formed, by the triple square of the two figures already found; the quotient, if it is not too great, ought to be such, that in writing it on the right of the two first figures of the root, and raising the number which results from it, to the cube, we may subtract this product from the three first portions; this new subtraction being made, bring down to the remainder the first figure of the fourth portion, and continue the same series of operations till you have brought down all the portions.

Remark. Frequently, in the course of operations, we may imagine that one of the quotients of which we have just spoken, is much too large, and that we may diminish it by two or more units; but by raising to the cube the root already found, together with this figure, if we subtract this cube from the amount of the portions already considered in the given number and obtain a very great remainder, we are led to think that the last figure obtained in the root is too small. We are guided (157) by this sign; *that the remainder equals or exceeds the triple square of the root obtained plus three times the same root, plus one.* In this case, we increase the root by one or several units of the order of the last figure obtained.

The following examples are for practice.

$$\sqrt[3]{483249} = 78 \text{ with a remainder } 8697;$$

$$\sqrt[3]{91632508641} = 4508 \text{ with a remainder } 20644129;$$

$$\sqrt[3]{32977340218432} = 32068 \text{ exactly.}$$

160. *Process for extracting the root of the n^{th} degree of a whole number.* In order to generalise the process of the extraction of roots, let us designate by N the proposed number, and by n the degree of the root to be extracted; if N has not more than n figures, its root has only a single figure, and we may obtain it immediately, by forming the n^{th} power of each of the whole numbers from 1 to 10 inclusive, of which the n^{th} power is 10^n , or the smallest number of $n + 1$ figures.

If N is composed of more than n figures, its root has more than one figure and may then be regarded as containing tens and units. Designating its tens by a and its units by b , we have (151)

$$N = (a + b)^n = a^n + n a^{n-1} b + n \frac{n-1}{2} a^{n-2} b^2 + \&c. ;$$

that is, the proposed number contains the n^{th} power of the tens, plus n times the product of the $n - 1$ power of the tens multiplied by the units, and a series of other parts which it is unnecessary to take account of.

Now the n^{th} power of the tens not giving units of an order inferior to unity, followed by n zeros, the last n figures on the right do not enter into this power. We must then separate them and extract the root of the greatest n^{th} power, contained in the portion on the left; this root will express the tens of the root sought.

If this portion on the left contained still more than n figures, we should separate the last n figures on the right, and extract the root of the greatest n^{th} power contained in the new portion on the left, and so on.

After having separated in this manner, the number N into portions of n figures (the last portion on the left having only n figures at the most), we extract the root of the greatest n^{th} power contained in that portion on the left, which gives the figure of the greatest units of the whole root, or the figure of the tens of the root of the number formed by the two first portions on the left. Subtracting the n^{th} power of this figure, from the first portion on the left, we obtain a remainder, which, followed by the second portion, contains still n times the product of the $n - 1$ power of the figure, by the following figure, and a series of other parts. But this first part evidently cannot give units of an order inferior to 10^{n-1} ; thus, the last $n - 1$ figures of the second portion cannot enter into this portion. It is sufficient then to bring down to the remainder corresponding to the first portion, the first figure of the second portion; and if, after having formed n times the $n - 1$ power of the first figure of the root, we divide by this result, the remainder followed by the first figure of the second portion, the quotient will express the second figure of the root, or will be a figure too great. In order to make trial of it, we write it on the right of the first figure, and raise the two figures to the n^{th} power; then we subtract the product obtained from the two first portions, which will give a new remain-

der, to which we bring down the first figure of the third portion, we then divide the number thus formed by n times the $n - 1$ power of the two figures of the root already found.

We may continue this series of operations till we have brought down all the portions.

The learner who has well understood the preceding steps may apply them to the extraction of the 4th, 5th, and 6th roots.

161. *Remark.* Whenever the number expressing the degree of the root to be extracted, is a multiple of two or several others, as 4, 6 . . . , the root may be obtained by a series of extractions of roots of a more simple degree. To illustrate these simplifications, we remark, that

$$(a^3)^4 = a^3 \times a^3 \times a^3 \times a^3 = a^{3+3+3+3} = a^{3 \times 4} = a^{12},$$

and that, in general,

$$(a^m)^n = a^m \times a^m \times a^m \times a^m \dots = a^{m \times n} \quad (16).$$

Then, the n^{th} power of the m^{th} power of a number is equal to the $m n^{\text{th}}$ power of that number.

Reciprocally, the $m n^{\text{th}}$ root of a number is equal to the n^{th} root of the m^{th} root of that number, or algebraically, we have

$$\sqrt[mn]{a} = \sqrt[n]{\sqrt[m]{a}}.$$

For, let there be

$$\sqrt[n]{\sqrt[m]{a}} = a';$$

we raise the two members to the n^{th} power; it becomes

$$\sqrt[m]{a} = a'^n;$$

[for, according to the definition of a root, we have

$$(\sqrt[n]{K})^n = K].$$

Let us raise again the two members to the m^{th} power; we obtain

$$a = (a'^n)^m = a'^{mn}.$$

Extracting the $m n^{\text{th}}$ root of the two members,

$$\sqrt[mn]{a} = a';$$

but we have already

$$\sqrt[n]{\sqrt[m]{a}} = a'; \text{ then } \sqrt[mn]{a} = \sqrt[n]{\sqrt[m]{a}}.$$

We shall find, according to this principle,

$$\sqrt[4]{256} = \sqrt{\sqrt{256}} = \sqrt{16} = 4;$$

$$\sqrt[6]{2985984} = \sqrt[3]{\sqrt{2985984}} = \sqrt[3]{1728} = 12;$$

$$\sqrt[6]{1771561} = \sqrt[3]{\sqrt{1771561}} = 11;$$

$$\sqrt[8]{1679616} = \sqrt[4]{1296} = \sqrt{\sqrt{1296}} = 6.$$

N. B. Although the successive roots may be extracted in any order whatever, it is preferable to extract first the root of the most simple degree, because then the extraction of the root of the highest degree, which is a more complicated operation, relates to a number having less figures than the proposed number. This is done in the second and third of the foregoing examples.

Of the Extraction of Roots by Approximation.

162. When the whole number of which the n^{th} root is required, is not a *perfect power*, the process of art. 160, gives only the whole portion of the root, or the root within unity. As to the fraction which is to complete the root, it cannot be obtained exactly; for the n^{th} power of an irreducible fractional number $\frac{a}{b}$ being $\frac{a^n}{b^n}$, this new number cannot be reduced to a whole number. But we can determine the root to any degree of approximation we wish.

Let it be, in general, proposed to extract the n^{th} root of a whole number a , to within a fraction $\frac{1}{p}$; that is, in such a manner that the error may be less than $\frac{1}{p}$.

Let us observe that a may be expressed under the form $\frac{a \times p^n}{p^n}$. If we designate by r the root of $a p^n$, obtained, to within unity, the number $\frac{a \times p^n}{p^n}$ or a is then comprehended between

$$\frac{r^n}{p^n} \text{ and } \frac{(r+1)^n}{p^n};$$

then also $\sqrt[n]{a}$ is comprehended between the roots of these two last

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numbers, that is, between

$$\frac{r}{p} \text{ and } \frac{r+1}{p}.$$

Therefore, $\frac{r}{p}$ is the required root, to within the fraction, $\frac{1}{p}$.

General Rule. *In order to extract the n^{th} root of a whole number, to within a fraction $\frac{1}{p}$, multiply the number by p^n ; extract the n^{th} root of the product, to within an unit, then divide the result by p .*

163. In the second place, let $\frac{a}{b}$ be a fraction or a fractional number of which we wish to extract the n^{th} root.

Let us multiply the two terms of the fraction by b^{n-1} ; we have

$$\frac{a}{b} = \frac{a b^{n-1}}{b^n}.$$

Let r be the n^{th} root of $a b^{n-1}$, to within unity;

$$\frac{a b^{n-1}}{b^n}, \text{ or } \frac{a}{b},$$

is comprehended between $\frac{r^n}{b^n}$ and $\frac{(r+1)^n}{b^n}$; then, $\sqrt[n]{\frac{a}{b}}$ is itself comprehended between $\frac{r}{b}$ and $\frac{r+1}{b}$. Thus, *after having rendered the denominator of the fraction a perfect n power, we extract the n^{th} root of the numerator, to within unity, then divide the result by the root of the new denominator.*

If we wish to have a greater degree of approximation than that indicated by the fraction $\frac{1}{b}$, we may extract the root of $a b^{n-1}$, to within any fraction whatever, $\frac{1}{p}$; let $r' + \frac{m}{p}$ be that root, then $r' + \frac{m}{p}$ will designate the required root to within a fraction indicated by $\frac{1}{p b}$.

Such are the *principles* of the extraction of roots by approximation, which we are about to apply to the particular case of the cube root, since that is the operation most frequently used.

164. Let it be proposed to extract the third root of 15, to within $\frac{1}{12}$.

We have

$$15 \times 12^3 = 15 \times 1728 = 25920.$$

Now the cube root of 25920, to within unity, is 29; then the required root is $2\frac{2}{3}$ or $2\frac{1}{3}$. (See article 162.)

Let it be proposed to extract the third root of 47 to within $\frac{1}{10}$.

We have

$$47 \times 20^3 = 47 \times 8000 = 376000.$$

Now the cube root of 376000 to within an unit, is 72; therefore,

$$\sqrt[3]{47} = \frac{72}{10} = 3\frac{1}{2} \text{ within } \frac{1}{10}.$$

Let it be proposed to find the value of $\sqrt[3]{25}$ within 0,001.

We must multiply 25 by the cube of 1000, or 1000000000, which gives 25000000000. Now the cube root of this number is 2920; then $\sqrt[3]{25} = 2,920$ within 0,001. (See article 162.)

In general, *to extract the third root of a whole number within a decimal unit, we must add three times as many zeros to the right of the number, as we wish to have decimal figures in the root; extract the root of the new number within an unit, then separate towards the right of that root the number of decimal figures required.*

165. Let it now be proposed to extract the cube root of a decimal fraction. Let us seek, for example, the cube root of $3,1415$. As the denominator 10000 of this fraction is not a perfect cube we must render it such by multiplying it by 100, which amounts to adding 2 zeros to the right of the proposed decimal fraction, and we have 3,141500. We next extract (163) the cube root, within unity, of 3141500, that is, of the number, leaving out the decimal point, which gives 146; then we divide the result by 100 or $\sqrt[3]{1000000}$; and we find $\sqrt[3]{3,1415} = 1,46$ within 0,01.

If we wish a greater degree of approximation, we add three times as many zeros more to the number, as we wish the number of decimal figures in the root to be increased.

In order to extract the cube root of a vulgar fraction to within a decimal unit, the most simple method consists in *reducing the proposed number to a decimal fraction and continuing the operation until we obtain three times the number of decimal figures which we wish the root to contain.* The question is then reduced to extracting the cube root of a decimal fraction.

166. Lastly, let it be proposed to extract the sixth root of 23 to within 0,01.

By applying to this example the rule given in article 162, we must multiply 23 by 100^6 , or add to the right of 23 twelve zeros, and then extract the sixth root of the result, to within unity, and divide this root by 100, or separate two decimal figures towards the right.

But we have (161)

$$\sqrt[6]{23 \times (100)^6} = \sqrt[3]{\sqrt{23 \times (100)^6}};$$

so that after having subtracted the square root of $23 \times (100)^6$ to within unity, we must extract the cube root of the result, and then divide the new result by 100 or separate two decimal figures towards the right.

We shall find by this method $\sqrt[6]{23} = 1,68$ to within 0,01.

The following examples will serve as an exercise;

$$\sqrt[3]{473} \text{ to within } \frac{1}{2^{\frac{1}{5}}} = \frac{1,5}{2^{\frac{1}{5}}}; \quad \sqrt[3]{79} \text{ to within } 0,0001 = 4,2908;$$

$$\sqrt[6]{13} \text{ to within } 0,01 = 1,53;$$

$$\sqrt[3]{3,00415} \text{ to within } 0,0001 = 1,4429;$$

$$\sqrt[3]{0,00101} \text{ to within } 0,01 = 0,10; \quad \sqrt[3]{\frac{1}{1\frac{1}{2}}} \text{ to within } 0,001 = 0,824.$$

III. *Formation of the Powers and the Extraction of the Roots of Algebraic Quantities. Calculus of Radicals.*

Let us first consider simple quantities.

167. Let it be required to form the fifth power of $2 a^3 b^2$; we have (2)

$$(2 a^3 b^2)^5 = 2 a^3 b^2 \times 2 a^3 b^2 \times 2 a^3 b^2 \times 2 a^3 b^2 \times 2 a^3 b^2;$$

whence we see that the coefficient 2 ought to be multiplied four times by itself, or that it should be raised to the fifth power; (2.) that each of the exponents of the letters should be added to itself four times or multiplied by 5.

Therefore,

$$(2 a^3 b^2)^5 = 2^5 \cdot a^{3 \times 5} b^{2 \times 5} = 32 a^{15} b^{10}.$$

In like manner,

$$(8 a^2 b^3 c)^3 = 8^3 \cdot a^{2 \times 3} b^{3 \times 3} c^3 = 512 a^6 b^9 c^3.$$

Thus, in order to raise a simple quantity to any given power, we must raise the coefficient to this power, and then multiply the exponents of the letters by the exponent of the power.

Reciprocally, in order to extract the root of a simple quantity to any degree whatever, we must (1.) extract the root of the coefficient, (2.) divide the exponent of each letter by the index of the root.

Thus,

$$\sqrt[3]{64 a^3 b^3 c^3} = 4 a b c; \sqrt[4]{16 a^8 b^{12} c^4} = 2 a^2 b^3 c.$$

We see, according to this rule, that in order that the quantity may be a perfect power of the degree of the root to be extracted, its coefficient must be a perfect power of this degree, and the exponents of its letters must be divisible by the exponent or index of the root to be extracted. We shall presently see how we can simplify the expression of the root of a quantity which is not a perfect power.

168. Hitherto we have paid no regard to the sign with which the simple quantity may be affected; but if we observe that whatever be the sign of the simple quantity, *its square is always positive*, and that every power of the degree designated by $2n$ may be considered equal to the n^{th} power of the square, that is, that $a^{2n} = (a^2)^n$, we conclude, that *every power of a quantity, either positive or negative, of a degree indicated by an even number is essentially positive*.

$$\text{Thus } (\pm 2 a^2 b^3 c)^4 = + 16 a^8 b^{12} c^4.$$

Moreover, as every power of a degree indicated by an odd number $2n + 1$, is the product of the power of the $2n$ degree by the first power, it follows that *every power of a degree indicated by an odd number is affected with the same sign as the quantity itself*.

$$\text{Then } (+ 4 a^2 b)^3 = + 64 a^6 b^3; (- 4 a^2 b)^3 = - 64 a^6 b^3.$$

It is evident from this, (1.) *that every root of a simple quantity, of a degree indicated by an odd number, must be affected with the same sign as the quantity*.

Thus,

$$\sqrt[3]{+ 8 a^3} = + 2 a; \sqrt[3]{- 8 a^3} = - 2 a; \sqrt[5]{- 32 a^{10} b^5} = - 2 a^2 b.$$

(2.) *That every root of a positive simple quantity, of a degree indicated by an even number, may have indifferently the sign + or -.*

Thus,

$$\sqrt[4]{81 a^4 b^{12}} = \pm 3 a b^3; \sqrt[6]{64 a^{18}} = \pm 2 a^3.$$

(3.) That every root of a negative simple quantity, of a degree indicated by an even number, is an impossible root; since there is no quantity which when raised to a power whose exponent is an even number, can give a negative result. So that

$$\sqrt[4]{-a}, \sqrt[6]{-b}, \sqrt[8]{-c},$$

are the symbols of operations which cannot be performed; these are *imaginary expressions* like $\sqrt{-a}$, $\sqrt{-b}$, (85).

169. We have already seen how we may raise a binomial $x + a$ to a power of any degree whatever; but it may happen that the terms of the binomial are affected with coefficients and exponents.

Let it be proposed, for example, to develope $(2 a^2 + 3 a b^3)$; let us put, for the moment,

$$2 a^2 = x, 3 a b = y;$$

we have

$$(2 a^2 + 3 a b)^3 = (x + y)^3 = x^3 + 3 x^2 y + 3 x y^2 + y^3.$$

Restoring $2 a^2$ and $3 a b$ in the place of x and y ,

$$(2 a^2 + 3 a b)^3 = (2 a^2)^3 + 3(2 a^2)^2 \cdot (3 a b) + 3(2 a^2) \cdot (3 a b)^2 + (3 a b)^3;$$

or, by performing the operations according to the rule given in article 167, and that for the multiplication of simple quantities,

$$(2 a^2 + 3 a b)^3 = 8 a^6 + 36 a^5 b + 54 a^4 b^2 + 27 a^3 b^3;$$

we shall find, in like manner,

$$\begin{aligned} (4 a^2 b - 3 a b c)^4 &= (x + y)^4 = x^4 + 4 x^3 y + 6 x^2 y^2 + 4 x y^3 + y^4 \\ &= (4 a^2 b)^4 + 4(4 a^2 b)^3 (-3 a b c) + 6(4 a^2 b)^2 (-3 a b c)^2 \\ &\quad + 4(4 a^2 b) (-3 a b c)^3 + (-3 a b c)^4 \\ &= 256 a^8 b^4 - 768 a^7 b^4 c + 864 a^6 b^4 c^2 - 432 a^5 b^4 c^3 + 81 a^4 b^4 c^4. \end{aligned}$$

(The signs are alternately positive and negative.)

Let it now be proposed to develope $(x + y + z)^3$; first, let us put $x + y = u$; it becomes

$$(u + z)^3 = u^3 + 3 z u^2 + 3 z^2 u + z^3;$$

or, by replacing $x + y$ instead of u ,

$$(x + y + z)^3 = (x + y)^3 + 3 z (x + y)^2 + 3 z^2 (x + y) + z^3,$$

or, by developing anew the operations indicated,

$$(x + y + z)^3 = x^3 + 3x^2y + 3xy^2 + y^3 + 3x^2z + 6xyz + 3y^2z + 3xz^2 + 3yz^2 + z^3,$$

This expression is composed of the *cubes of the three terms, plus three times the square of each term multiplied by the first power of the other two, plus six times the product of the three terms.*

This law is easily verified in the case of any polynomial (86).

In order to apply the preceding formula to the développement of the cube of a trinomial whose terms have coefficients and exponents, we must, as in the case of binomials, *designate each term by a single letter, develop, and then restore the terms in the place of the letters introduced, and perform the operations indicated.*

We shall thus find, all the operations being performed,

$$(2a^2 - 4ab + 3b^2)^3 = 8a^6 - 48a^5b + 132a^4b^2 - 208a^3b^3 + 198a^2b^4 - 108ab^5 + 27b^6.$$

We may develop by an analogous process, the fourth, fifth, and in general, any power of any polynomial.

170. With respect to the extraction of the roots of polynomials, we shall confine ourselves to explaining the process for the cube root, it will be easy afterwards to generalize.

Let N be the given polynomial, R its cube root. Let us suppose, as in the case of the square root (87), that these two polynomials are arranged with reference to a certain letter, a for example. According to the law for the composition of the cube of a polynomial, the cube of R contains two parts which cannot be reduced with the others; these are the cube of the first term and three times the square of the first by the second; for it is evident, that these two parts contain the letter a with an exponent greater than in the triple product of the square of the second term by the first, in the cube of the second term, in the triple product of the square of the first by the third, &c. Then these two parts form necessarily the first and second term of N . Thus, *by extracting the cube root of the first term of N , we shall have the first term of R , then by dividing the second term of N by three times the square of the first term of R , (the triple square is easily formed) we shall have the second term of R .* By knowing the two first terms of R , we form the cube of this binomial, and subtract it from N . The remainder N' contains still the product of the triple square of the

first term of R by the third, plus a series of other parts containing a with an exponent less than that which enters into this product, and which is consequently the first term of the remainder N' . Then, if we divide the first term of N' by three times the square of the first term of R , we shall necessarily have the third term of R . By forming the cube of the trinomial already found for the root, and subtracting this cube from N , we shall have a new remainder N'' , upon which we may perform the same operation as upon the remainder N' , and so on.

If now we bring together the parts of the preceding reasoning, put in *italics*, we may deduce a general method for the extraction of the cube root of any polynomial whatever, and we can apply it to the polynomials developed in article 169.

We shall content ourselves with indicating to those who wish to discover the process for the extraction of the n^{th} root, that if we designate the different terms of the root by $x + y + z + u \dots$ the expression for the n^{th} power contains among other parts,

$$x^n + n x^{n-1} y \dots + n x^{n-1} z + \dots + n x^{n-1} u ;$$

and it is these parts which, presented successively in the proposed polynomial, serve to find the different terms of the root.

Calculus of Radicals.

171. When the quantity either simple or polynomial, whose root, of a certain degree, is required, is not a perfect power, we can only indicate the operation (2) by placing the sign $\sqrt{}$ before the quantity, and above this sign the number which indicates the degree of the root to be extracted. This number is called the *index of the radical*.

We can frequently subject *radical expressions* to certain simplifications, founded upon a principle analogous to that of article 84, namely, that *the root of the n^{th} degree of a product is equal to the product of the n roots of its different factors*.

In algebraic terms,

$$\sqrt[n]{abcd \dots} = \sqrt[n]{a} \times \sqrt[n]{b} \times \sqrt[n]{c} \times \sqrt[n]{d} \dots$$

For, if we raise each of these two expressions to the n^{th} power, we shall find for the first

$$\sqrt[n]{abc\dots})^n = abc\dots,$$

and for the second,

$$(\sqrt[n]{a} \times \sqrt[n]{b} \times \sqrt[n]{c}\dots)^n = (\sqrt[n]{a})^n \cdot (\sqrt[n]{b})^n \cdot (\sqrt[n]{c})^n \dots = abc\dots$$

Then, since the n powers of the expressions are equal, they must themselves be equal (176).

This being premised, let there be the expression $\sqrt[3]{54a^4b^3c^3}$, which cannot be replaced by a simple rational quantity, since 54 is not a perfect cube, and moreover, because the exponents of a and c are not divisible by 3, we have

$$\sqrt[3]{54a^4b^3c^3} = \sqrt[3]{27a^3b^3} \cdot \sqrt[3]{2ac^3} = 3ab\sqrt[3]{2ac^3}.$$

And, in the same manner,

$$\sqrt[3]{8a^3} = 2\sqrt[3]{a^3}; \quad \sqrt[4]{48a^5b^3c^6} = 2ab^3c\sqrt[4]{3ac^3};$$

$$\sqrt[6]{192a^7b^3c^{12}} = \sqrt[6]{64a^6c^{12}} \times \sqrt[6]{3ab} = 2a^2c^2\sqrt[6]{3ab}.$$

In the expressions

$$3ab\sqrt[3]{2ac^3}, \quad 2\sqrt[3]{a^3}, \quad 2ab^3c\sqrt[4]{3ac^3},$$

the quantities which precede the radical, *with the sign of multiplication*, are called *coefficients* of the radical.

172. The principle demonstrated in article 161, affords another kind of simplification.

If we have, for example, the radical expression $\sqrt[6]{4a^2}$; since by this principle

$$\sqrt[6]{4a^2} = \sqrt[3]{\sqrt[4]{4a^2}},$$

and as the quantity under the radical $\sqrt[4]{}$ is a perfect square, we may perform this extraction of the square root, which gives

$$\sqrt[6]{4a^2} = \sqrt[3]{2a}.$$

And, in the same manner,

$$\sqrt[4]{36a^2b^3} = \sqrt{\sqrt[4]{36a^2b^3}} = \sqrt{6ab}.$$

In general,

$$\sqrt[m]{a^n} = \sqrt[n]{\sqrt[m]{a^n}} = \sqrt[n]{a},$$

that is, when the index of the radical is a multiple of a certain number n , and the quantity under the radical sign is an exact power of the degree indicated by n , we may, without changing the value of the radical, divide its index by n , and extract the n^{th} root of the quantity under the sign.

The above proposition is the reverse of one not less important, which is this, that we may multiply the index of a radical by a certain number, provided that we raise the quantity under the sign to the power indicated by this number.

Thus, $\sqrt[m]{a} = \sqrt[mn]{a^n}$. For a is the same as $\sqrt[n]{a^n}$; then

$$\sqrt[m]{a} = \sqrt[m]{\sqrt[n]{a^n}} = \sqrt[mn]{a^n}.$$

This last principle serves to reduce two or more radicals to the same index, which is frequently very useful.

For example, let there be the two radical expressions

$$= \sqrt[3]{2a} \text{ and } \sqrt[4]{(a+b)},$$

which we wish to reduce to the same index.

If we multiply the index of the first by 4, the index of the second, and raise the quantity $2a$ to the fourth power; if, also, we multiply the index of the second by 3, the index of the first, and raise the quantity $(a+b)$ to the third power, we do not change the values of the two radicals, and we obtain by these operations

$$\sqrt[3]{2a} = \sqrt[12]{2^4 a^4} = \sqrt[12]{16 a^4}; \quad \sqrt[4]{a+b} = \sqrt[12]{(a+b)^3}.$$

General Rule. *In order to reduce two or more radicals to the same index, multiply the index of each radical by the product of all the other indices, and raise the quantity under the radical sign to the power indicated by this product.*

This rule, which is very similar to that of the reduction of fractions to a common denominator, is susceptible of the same modifications.

Let there be, for example, the radicals

$$\sqrt[4]{a}, \sqrt[6]{5b}, \sqrt[8]{a^2 + b^2},$$

to be reduced to the same index.

Since the numbers 4, 6, 8, have common factors, and since 24 is the most simple multiple of these numbers, it is evidently sufficient to multiply the first by 6, the second by 4, and the third by

3, provided that we raise the quantities under the radical signs to the sixth, fourth, and third powers respectively, which gives

$$\sqrt[4]{a} = \sqrt[24]{a^6}; \sqrt[6]{b} = \sqrt[24]{b^4}; \sqrt[8]{a^3 + b^3} = \sqrt[24]{(a^3 + b^3)^3}.$$

These principles being established, we propose to perform upon radical quantities the operations of Arithmetic, which are six in number, including the formation of powers and the extraction of their roots.

173. Addition and Subtraction of Radicals. Two radicals are said to be similar when they have the same index, and when the quantity under the radical sign is likewise the same.

This being laid down, in order to add or subtract two similar radical quantities, we must add or subtract their coefficients, and put the sum or difference as a coefficient before the common radical.

Thus,

$$3\sqrt[3]{b} + 2\sqrt[3]{b} = 5\sqrt[3]{b}, \quad 3\sqrt[3]{b} - 2\sqrt[3]{b} = \sqrt[3]{b},$$

$$3a\sqrt[4]{b} \pm 2c\sqrt[4]{b} = (3a \pm 2c)\sqrt[4]{b}.$$

Frequently two radical quantities are not at first similar, but may be made so by applying to them the simplifications of articles 171 and 172. For example,

$$\begin{aligned} \sqrt[2]{48ab^3} + b\sqrt[2]{75a} &= 4b\sqrt[2]{3a} + 5b\sqrt[2]{3a} = 9b\sqrt[2]{3a}; \\ \sqrt[3]{8a^3b + 16a^4} - \sqrt[3]{b^4 + 2ab^3} &= 2a\sqrt[3]{b + 2a} - b\sqrt[3]{b + 2a} \\ &= (2a - b)\sqrt[3]{b + 2a}, \\ 3\sqrt[6]{4a^3} + 2\sqrt[3]{2a} &= 3\sqrt[3]{2a} + 2\sqrt[3]{2a} = 5\sqrt[3]{2a}. \end{aligned}$$

If the radicals are not similar, we can only indicate their addition or subtraction by means of the signs + and —.

174. Multiplication and Division. Let us first consider the case where the radicals have the same index.

Let it be required to multiply or divide $\sqrt[n]{a}$ by $\sqrt[n]{b}$, we say that we shall have

$$\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab} \text{ and } \sqrt[n]{a} \div \sqrt[n]{b} = \sqrt[n]{\frac{a}{b}}.$$

Thus,

$\sqrt[3]{\sqrt[4]{8a^3}}$ being equal to $\sqrt[4]{\sqrt[3]{8a^3}}$, it is reduced to $\sqrt[4]{2a}$.

In like manner,

$$\sqrt[2]{\sqrt[5]{9a^2}} = \sqrt[5]{\sqrt[2]{9a^2}} = \sqrt[5]{3a}.$$

It is moreover evident, that we have

$$\sqrt[n]{\sqrt[m]{a}} = \sqrt[m]{\sqrt[n]{a}};$$

since each of these expressions is equal to $\sqrt[nm]{a}$.

176. The rules which we have just established for the calculation of radicals are founded chiefly upon the principle, that the n^{th} root of the product of several factors is equal to the product of the n^{th} roots of these factors, and the demonstration of this principle depends upon this, *if the powers of the same degree of two expressions are equal, the expressions themselves are equal*. Now this last proposition, which is true when we consider only absolute numbers, is not always so for the different expressions to which Algebra may lead us.

In order to verify the truth of this assertion we will prove that the same number may have *algebraically* several *square roots*, several *cube roots*, several *fourth roots*, &c.

For, let us designate by x the general expression of the square root of a number a , and by p the *numerical or arithmetical value* of this square root, we have the equation

$$x^2 = a, \text{ or } x^2 = p^2,$$

from which we deduce $x = \pm p$. Whence we see that with whatever sign we affect the arithmetical value p of the square root of a , its square is equally a ; a result conformable to what has been said article 85.

In the second place, let x be the general expression for the cube root of a , and let us designate by p the numerical value of this cube root; we have the equation

$$x^3 = a, \text{ or } x^3 = p^3.$$

Now we observe that we may put $x^3 = p^3$ under the form $x^3 - p^3 = 0$.

We have seen (31) that the expression $x^3 - p^3$ is divisible by $x - p$ and gives for an exact quotient

$$x^2 + px + p^2;$$

the above equation can then be transformed thus,

$$(x - p)(x^2 + px + p^2) = 0,$$

an equation which may be satisfied either by putting

$$x - p = 0, \text{ whence } x = p,$$

or, by putting

$$x^2 + px + p^2 = 0,$$

whence

$$x = -\frac{p}{2} \pm \frac{p}{2} \sqrt{-3},$$

or

$$x = p \left(\frac{-1 \pm \sqrt{-3}}{2} \right).$$

We see then that the cube root of a admits of three different algebraic values, namely,

$$p, p \left(\frac{-1 + \sqrt{-3}}{2} \right) \text{ and } p \left(\frac{-1 - \sqrt{-3}}{2} \right).$$

Let it be further required to resolve the equation $x^4 = p^4$ (p designating the arithmetical value of the $\sqrt[4]{a}$). This equation may be put under the form $x^4 - p^4 = 0$. Now the expression $x^4 - p^4$ is reduced (19) to

$$(x^2 - p^2)(x^2 + p^2).$$

Then the equation itself is reduced to

$$(x^2 - p^2)(x^2 + p^2) = 0,$$

and we can satisfy it either by putting

$$x^2 - p^2 = 0; \text{ whence } x = \pm p,$$

or, by putting

$$x^2 + p^2 = 0; \text{ whence } x = \pm \sqrt{-p^2} = \pm p \sqrt{-1}.$$

We thus obtain four expressions algebraically different for the fourth root of a number a .

Let us propose to resolve the new equation $x^6 = p^6$, which may be put under the form $x^6 - p^6 = 0$.

Now, $x^6 - p^6$ is reduced (19) to

$$(x^3 - p^3)(x^3 + p^3) = 0.$$

Thus the equation becomes

$$(x^3 - p^3)(x^3 + p^3) = 0.$$

Already the equation $x^3 - p^3 = 0$, resolved in the preceding article, has given

$$x = p \text{ and } x = p \left(\frac{-1 \pm \sqrt{-3}}{2} \right).$$

Let us consider the equation $x^3 + p^3 = 0$, and observe that if we substitute for the present $-p'$ instead of p , it becomes

$$x^3 - p'^3 = 0;$$

whence we deduce

$$x = p', \text{ and } x = p' \left(\frac{-1 \pm \sqrt{-3}}{2} \right);$$

or, by restoring in the place of p' , its value $-p$,

$$x = -p, \text{ and } x = -p \left(\frac{-1 \pm \sqrt{-3}}{2} \right).$$

Thus, the equation $x^6 - p^6 = 0$, and consequently the sixth root of a admits of six values;

$$p, \alpha p, \alpha' p, -p, -\alpha p, -\alpha' p,$$

by putting, for the purpose of simplifying,

$$\alpha = \frac{-1 + \sqrt{-3}}{2}, \alpha' = \frac{-1 - \sqrt{-3}}{2}.$$

We may conclude from analogy (which, moreover, will be demonstrated hereafter in a more complete manner), that every equation of the form of

$$x^m - a = 0, \text{ or } x^m - p^m = 0,$$

is susceptible of m different values, that is, that the m^{th} root of a number admits of m values, algebraically different.

177. *Consequence.* If in the preceding equations and the results which correspond to them, we suppose, as a particular case, $a = 1$, which gives $p = 1$, we shall obtain the square roots, the cube roots, the fourth roots, &c., of unity. Thus, $+1$ and -1 are the two square roots of unity, since the equation $x^2 - 1 = 0$ gives $x = \pm 1$.

In like manner,

$$+1, \frac{-1 + \sqrt{-3}}{2}, \frac{-1 - \sqrt{-3}}{2},$$

are the three cube roots of unity, or the roots of $x^3 - 1 = 0$.

$$+ 1, - 1, + \sqrt{-1}, - \sqrt{-1},$$

are the four fourth roots of unity, or the roots of $x^4 - 1 = 0$.

178. It follows from the preceding analysis, that the rules for the calculus of radical expressions, which are exact, so long as we confine the operations to absolute numbers, may be susceptible of certain *modifications*, when the operations are performed upon *expressions or symbols purely algebraical*, and especially when we apply these rules to imaginary expressions, these modifications are necessary, and they are a consequence of what has been said in article 176.

For example, if we demand the product of $\sqrt{-a}$ by $\sqrt{-a}$; the rule of article 174 gives

$$\sqrt{-a} \times \sqrt{-a} = \sqrt{+a^2}.$$

Now, the $\sqrt{a^2}$ is equal to $\pm a$ (176). There is here then apparently a doubt as it regards the sign with which a should be affected in order to answer the question. Still, the true answer is $-a$, for the reason, that in general in order to raise $\sqrt{m}$ to the square, it is sufficient to suppress the radical sign; now, $\sqrt{-a} \times \sqrt{-a}$ is reduced to $(\sqrt{-a})^2$, and, consequently, is equal to $-a$.

In the second place, let it be required to form the product

$$\sqrt{-a} \times \sqrt{-b};$$

we shall have, according to the rule of article 174,

$$\sqrt{-a} \times \sqrt{-b} = \sqrt{+ab}.$$

Now, $\sqrt{ab} = \pm p$ (176), p designating the arithmetical value of the square root of ab ; but we say, that the true result ought to be $-p$, or $-\sqrt{ab}$, so long as we consider each of the two radicals $\sqrt{-a}$ and $\sqrt{-b}$, as preceded by the sign $+$.

For we have

$$\sqrt{-a} = \sqrt{a} \cdot \sqrt{-1}, \text{ and } \sqrt{-b} = \sqrt{b} \cdot \sqrt{-1};$$

therefore,

$$\begin{aligned} \sqrt{-a} \times \sqrt{-b} &= \sqrt{a} \cdot \sqrt{-1} \times \sqrt{b} \cdot \sqrt{-1} = \sqrt{ab} \cdot (\sqrt{-1})^2 \\ &= \sqrt{ab} \times -1 = -\sqrt{ab}. \end{aligned}$$

We shall find, according to these principles, for the different powers of $\sqrt{-1}$,

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$$\sqrt{-1} = \sqrt{-1}, (\sqrt{-1})^2 = -1,$$

$$(\sqrt{-1})^3 = (\sqrt{-1})^2 \cdot \sqrt{-1} = -\sqrt{-1},$$

and

$$(\sqrt{-1})^4 = (\sqrt{-1})^2 \cdot (\sqrt{-1})^2 = -1 \times -1 = +1.$$

As the four following powers are obtained by multiplying the fourth $+1$, by the first, second, third, and fourth, we shall still find for these four new powers,

$$+ \sqrt{-1}, -1, -\sqrt{-1} + 1;$$

therefore, in general, all the powers of $\sqrt{-1}$, taken *four and four*, form a period equal to

$$\sqrt{-1}, -1, -\sqrt{-1}, +1.$$

Let it be further proposed to determine the product of $\sqrt[4]{-a}$ by $\sqrt[4]{-b}$, which, according to the rule, will be $\sqrt[4]{+ab}$, and, consequently, will give the four values,

$$+ \sqrt[4]{ab}, -\sqrt[4]{ab}, + \sqrt[4]{ab} \cdot \sqrt{-1}, -\sqrt[4]{ab} \cdot \sqrt{-1}.$$

In order to determine the true product, we observe that

$$\sqrt[4]{-a} = \sqrt[4]{a} \cdot \sqrt{-1}, \sqrt[4]{-b} = \sqrt[4]{b} \cdot \sqrt{-1};$$

but

$$\sqrt{-1} \times \sqrt{-1} = (\sqrt{-1})^2 = (\sqrt{\sqrt{-1}})^2 = \sqrt{-1};$$

then

$$\sqrt[4]{-a} \times \sqrt[4]{-b} = \sqrt[4]{ab} \cdot \sqrt{-1}.$$

Let us apply the preceding calculations to the verification of the expression

$$\frac{-1 + \sqrt{-3}}{2},$$

as a root of the equation $x^3 - 1 = 0$; that is, as a *cube root* of 1 (177). According to the formula

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$

we have

$$\begin{aligned} & \left(\frac{-1 + \sqrt{-3}}{2} \right)^3 \\ = & \frac{(-1)^3 + 3(-1)^2 \cdot \sqrt{-3} + 3(-1) \cdot (\sqrt{-3})^2 + (\sqrt{-3})^3}{8} \end{aligned}$$

$$= \frac{-1 + 3\sqrt{-3} - 3 \times -3 - 3\sqrt{-3}}{8} = \frac{8}{8} = 1.$$

We may verify, in the same manner, the second value,

$$\frac{-1 - \sqrt{-3}}{2}.$$

IV. Theory of Exponents of whatever Nature.

179. We must here make known two new kinds of notation of great use in algebraic calculations, namely, fractional exponents and negative exponents; they derive their origin from the rules established for the extraction of roots and the division of simple quantities.

Let it be required to extract the n^{th} root of such a quantity as a^m . We have seen (167) that if m is a multiple of n , we must divide the exponent m by the index n of the root. But if m is not divisible by n , in which case the extraction of the root is not possible algebraically, we may agree to indicate this operation by indicating the division of the two exponents.

Then $\sqrt[n]{a^m} = a^{\frac{m}{n}}$, according to the convention founded upon the rule for the exponents in the extraction of the roots of simple quantities.

Thus,

$$\sqrt[3]{a^2} = a^{\frac{2}{3}}; \sqrt[4]{a^7} = a^{\frac{7}{4}}.$$

In like manner, if it is required to divide a^m by a^n , we know (23) that we must subtract the exponent of the divisor from that of the dividend whenever we have $m > n$, which gives

$$\frac{a^m}{a^n} = a^{m-n}.$$

But if $m < n$, in which case the division is impossible algebraically, we may agree to indicate this division by always subtracting the exponent of the divisor from that of the dividend. Let p be the absolute difference between n and m , we have then

$$n = m + p, \text{ whence } \frac{a^m}{a^{m+p}} = a^{-p};$$

moreover, $\frac{a^m}{a^{m+p}}$ is reduced to $\frac{1}{a^p}$, by suppressing the factor a^m common to the two terms. Therefore, $a^{-p} = \frac{1}{a^p}$.

The expression a^{-p} is then the symbol of a division which cannot be performed, and its true value is the quotient arising from the division of unity by the same number a , with the exponent p taken positively. Thus,

$$a^{-3} = \frac{1}{a^3}; \quad a^{-5} = \frac{1}{a^5}.$$

The notation of negative exponents has the advantage of preserving an entire form to fractional expressions.

From combining the extraction of the root with the impossible division of simple quantities, there results another notation, that of *negative fractional exponents*.

Let it be required to extract the n^{th} root of $\frac{1}{a^m}$.

In the first place, we have

$$\frac{1}{a^m} = a^{-m}, \text{ then } \sqrt[n]{\frac{1}{a^m}} = \sqrt[n]{a^{-m}} = a^{-\frac{m}{n}},$$

by substituting for the ordinary radical sign a fractional exponent.

The expressions

$$\sqrt[n]{a^{\frac{m}{n}}}, a^{-p}, a^{-\frac{m}{n}},$$

are therefore, according to a convention founded upon the preceding established rules, notations equivalent to

$$\sqrt[n]{a^{\frac{m}{n}}}, \frac{1}{a^p}, \sqrt[n]{\frac{1}{a^{\frac{m}{n}}}}.$$

Thus, we may substitute the first of these expressions for the second, and the reverse, as circumstances render it convenient.

As in common language a^p is said to be the p power of a , p being a positive whole number, so also, by analogy,

$$\sqrt[n]{a^{\frac{m}{n}}}, a^{-p}, a^{-\frac{m}{n}},$$

are called the $\frac{m}{n}$ power, the $-p$ power, and the $-\frac{m}{n}$ power of a ; which has led algebraists to generalize the word *power*. But

it would perhaps be more proper to employ only the denominations a , with the exponent $\frac{m}{n}$, with the exponent $-p$, with the exponent $-\frac{m}{n}$, restricting the use of the word *power* to designating the product of a number multiplied several times by itself (2).

180. These ideas concerning the origin and signification of quantities affected by any exponents whatever, being established, let us see how we are to perform upon them the operations of Arithmetic, beginning with multiplication.

Multiplication. In the first place, let it be required to multiply $a^{\frac{2}{3}}$ by $a^{\frac{2}{3}}$. We say that it is sufficient to *add the two exponents*, and we shall have

$$a^{\frac{2}{3}} \times a^{\frac{2}{3}} = a^{\frac{2}{3} + \frac{2}{3}} = a^{\frac{4}{3}}.$$

For, we have seen (179) that

$$a^{\frac{2}{3}} = \sqrt[3]{a^2}; \quad a^{\frac{2}{3}} = \sqrt[3]{a^2};$$

then

$$a^{\frac{2}{3}} \times a^{\frac{2}{3}} = \sqrt[3]{a^2} \times \sqrt[3]{a^2};$$

or, by performing the multiplication, according to the rule of article 174,

$$a^{\frac{2}{3}} \times a^{\frac{2}{3}} = \sqrt[3]{a^4} = a^{\frac{4}{3}}.$$

Again, let it be required to multiply $a^{-\frac{3}{4}}$ by $a^{\frac{5}{8}}$, we say that we shall have

$$a^{-\frac{3}{4}} \times a^{\frac{5}{8}} = a^{-\frac{3}{4} + \frac{5}{8}} = a^{-\frac{6}{8} + \frac{5}{8}} = a^{-\frac{1}{8}};$$

for

$$a^{-\frac{3}{4}} = \sqrt[4]{\frac{1}{a^3}}, \quad a^{\frac{5}{8}} = \sqrt[8]{a^5};$$

therefore,

$$a^{-\frac{3}{4}} \times a^{\frac{5}{8}} = \sqrt[4]{\frac{1}{a^3}} \times \sqrt[8]{a^5} = \sqrt[8]{\frac{1}{a^6}} \times \sqrt[8]{a^5} = \sqrt[8]{\frac{a^5}{a^6}} = \sqrt[8]{\frac{1}{a}} = a^{-\frac{1}{8}}.$$

In general, let it be required to multiply $a^{-\frac{m}{n}}$ by $a^{\frac{p}{q}}$; we have

$$a^{-\frac{m}{n}} \times a^{\frac{p}{q}} = a^{-\frac{m}{n} + \frac{p}{q}} = a^{\frac{pq - mq}{nq}};$$

for $a^{-\frac{m}{n}} = \sqrt[n]{\frac{1}{a^m}}, a^{\frac{p}{q}} = \sqrt[q]{a^p};$

therefore

$$a^{-\frac{m}{n}} \times a^{\frac{p}{q}} = \sqrt[n]{\frac{1}{a^m}} \times \sqrt[q]{a^p} = \sqrt[nq]{\frac{a^{np}}{a^{mq}}} = \sqrt[nq]{a^{np-mq}} = a^{\frac{np-mq}{nq}}.$$

Thence the *general rule* for multiplying two simple quantities having any exponents whatever, is as follows; *we must add together the two exponents of the same letter*; this is the same with that established in article 16, for quantities having entire exponents.

We shall find, according to this rule,

$$a^{\frac{3}{4}} b^{-\frac{1}{2}} c^{-1} \times a^2 b^{\frac{2}{3}} c^{\frac{3}{5}} = a^{\frac{11}{4}} b^{\frac{1}{6}} c^{-\frac{2}{5}};$$

$$3 a^{-2} b^{\frac{2}{3}} \times 2 a^{-\frac{4}{5}} b^{\frac{1}{2}} c^2 = 6 a^{-\frac{14}{5}} b^{\frac{7}{6}} c^2.$$

Division. In order to divide the one by the other two simple quantities having any exponents whatever, we must follow the rule for quantities having entire and positive exponents, that is, for each letter *we must subtract the exponent of the divisor from that of the dividend*.

For, the exponent of each letter in the quotient must be such that when added to that of the same letter in the divisor, the sum shall be equal to the exponent of the dividend; then the exponent of the quotient is equal to the difference between the exponent of the dividend and that of the divisor.

We shall find, according to this rule,

$$a^{\frac{2}{3}} : a^{-\frac{3}{4}} = a^{\frac{2}{3} - (-\frac{3}{4})} = a^{\frac{17}{12}};$$

$$a^{\frac{3}{4}} : a^{\frac{4}{5}} = a^{\frac{3}{4} - \frac{4}{5}} = a^{-\frac{1}{20}}; a^{\frac{2}{3}} : a^{\frac{1}{2}} = a^{\frac{1}{6}};$$

$$a^{\frac{2}{3}} b^{\frac{3}{4}} : a^{-\frac{1}{2}} b^{\frac{7}{8}} = a^{\frac{9}{6}} b^{-\frac{1}{8}}.$$

Formation of Powers. In order to raise a simple quantity having any exponent whatever to the m^{th} power, conformably to the rule given in article 167, *we must multiply the exponent of each letter by the exponent m of the power*, since raising the quantity to the m^{th} power, is to multiply it $m - 1$ times by itself; then, according to the rule for multiplication, we must add the exponent of each letter $m - 1$ times to itself, or multiply it by m .

Thus,

$$\left(a^{\frac{2}{3}}\right)^5 = a^{\frac{10}{3}}; \left(a^{\frac{2}{3}}\right)^3 = a^2 = a^{\frac{6}{3}};$$

$$\left(2 a^{-\frac{1}{2}} b^{\frac{3}{4}}\right)^6 = 64 a^{-3} b^{\frac{9}{2}}; \left(a^{-\frac{5}{6}}\right)^{12} = a^{-10}.$$

Extraction of Roots. In order to extract the n^{th} root of a simple quantity, conformably to the rule in article 167, *we must divide the exponent of each letter by the index n of the root.*

For, the exponent of each letter in the result must be such, that, when multiplied by the index n of the root to be extracted, the product shall be the exponent which the letter has in the proposed simple quantity; therefore, the exponents in the result must be respectively equal to the quotients arising from the division of the exponents in the proposed simple quantity, by the index n of the root.

Thus,

$$\sqrt[3]{a^{\frac{2}{3}}} = a^{\frac{2}{9}}; \sqrt[4]{a^{\frac{8}{11}}} = a^{\frac{2}{11}}; \sqrt[2]{a^{-\frac{3}{4}}} = a^{-\frac{3}{8}};$$

$$\sqrt[3]{a^{\frac{2}{3}} b^{-2}} = a^{\frac{1}{3}} b^{-\frac{2}{3}} \dots$$

The last three rules have been easily deduced from the rule relative to multiplication; but they might be demonstrated by going back to the original quantities having any exponents whatever.

We conclude, by an operation equivalent to a demonstration, which embraces implicitly the two preceding.

Let it be required to raise $a^{\frac{m}{n}}$ to the $-\frac{r}{s}$ power, we must prove that we have

$$\left(a^{\frac{m}{n}}\right)^{-\frac{r}{s}} = a^{\frac{m}{n} \times -\frac{r}{s}} = a^{-\frac{mr}{ns}}.$$

For, if we go back to the origin of these notations, we find that

$$\begin{aligned} \left(a^{\frac{m}{n}}\right)^{-\frac{r}{s}} &= \sqrt[s]{\frac{1}{\left(a^{\frac{m}{n}}\right)^r}} = \sqrt[s]{\frac{1}{\left(\sqrt[n]{a^m}\right)^r}} = \sqrt[s]{\frac{1}{\sqrt[n]{a^{mr}}}} \\ &= \sqrt[s]{\sqrt[n]{\frac{1}{a^{mr}}}} = \sqrt[\frac{ns}{1}]{a^{-mr}} = a^{-\frac{mr}{ns}}. \end{aligned}$$

The advantage derived from the employment of exponents, of any nature whatever, consists principally in this, that the calculation of expressions of this kind requires no other rules than those which have been established for the calculation of quantities having entire exponents. Moreover these calculations are reduced to simple operations upon fractions, operations with which we are very familiar.

Of Progression by Differences and by Quotients.

Progression by Differences.

181. We give the name of *equidifference*, *arithmetical progression*, or *progression by differences* to a series of terms, each of which exceeds or is exceeded by that which precedes it, by a constant quantity which is termed the *ratio* or *difference* of the progression.

Thus, let there be the two series

$$\begin{array}{l} 1, \quad 4, \quad 7, \quad 10, \quad 13, \quad 16, \quad 19, \quad 22, \quad 25 \dots \\ 60, \quad 56, \quad 52, \quad 48, \quad 44, \quad 40, \quad 36, \quad 32, \quad 28 \dots \end{array}$$

The first is called an *increasing progression*, of which the *ratio* is 3, and the second a *decreasing progression*, of which the *ratio* is 4.

Let us designate, in general, by $a, b, c, d, e, f \dots$ the terms of a progression by differences; it is usually expressed thus;

$$\div a . b . c . d . e . f . g . h . i . k \dots$$

and we read it in this manner,

$$a \text{ is to } b, \text{ as } b \text{ is to } c, \text{ as } c \text{ is to } d, \text{ as } d \text{ is to } e \dots,$$

or, more concisely,

$$a \text{ is to } b \text{ is to } c \text{ is to } d \text{ is to } e \dots$$

This is a series of continued equidifferences, in which each term is at the same time both consequent and antecedent, with the exception of the first term, which is only an antecedent, and the last, which is only a consequent.

182. Let us call r the *ratio* of the progression, which we will suppose increasing, in all which is to follow. (If it were decreasing, it would be sufficient to change r to $-r$ in the results.)

This being laid down, we have evidently, according to the definition of the progression,

$$b = a + r, c = b + r = a + 2r, d = c + r = a + 3r \dots;$$

and, in general, a term of any place whatever is equal to the first, plus as many times the ratio as there are terms before that under consideration. Thus, let l be this term, and n the whole number of terms to this, inclusive; we have for the expression of this general term,

$$l = a + (n - 1)r.$$

For, if we suppose $n = 1, 2, 3, 4, \dots$ successively, we shall find the first, second, third, $\dots$ term of the progression.

If the progression were decreasing, we should have, on the other hand,

$$l = a - (n - 1)r.$$

The formula, $l = a + (n - 1)r$, serves to give the expression of a term of any place whatever, without being obliged to determine all those which precede it.

Thus, if we seek the 50th term of the progression

$$\div 1.4.7.10.13.16.19 \dots$$

we have, by making $n = 50$,

$$l = 1 + 49.3 = 148.$$

183. A progression by differences being given, it may be proposed to determine the sum of a certain number of terms.

Let there be the progression

$$\div a.b.c.d.e.f \dots i.k.l,$$

continued to the term l inclusive, and let n be the number of terms, and r the ratio.

We begin by observing, that if x designates a term which has p terms before it, and y a term which has p terms after it, we have, according to what has just been said, the equalities

$$x = a + p \times r,$$

$$y = l - p \times r;$$

whence we deduce, by adding

$$x + y = a + l;$$

which shows, that in every progression, the sum of any two terms, taken at equal distances from the extremes, is equal to the sum of the extremes; or the two extremes, and two terms taken at equal distances from these extremes, form an equidifference (in the order in which they are written.)

This being admitted, let us suppose that we have written the progression under itself, but in an inverse order, thus

$$\begin{array}{c} \div a . b . c . d . e . f i . k . l , \\ \div l . k . i c . b . a . \end{array}$$

Let us call S the sum of the terms of the first progression, $2S$ will be the sum of the terms of the two progressions; and we shall have, by uniting the terms in the same vertical column,

$$2S = (a+l) + (b+k) + (c+i) + \dots + (i+c) + (k+b) + (l+a);$$

or indeed, since all the parts $a+l$, $b+k$, $c+i$, ..., are equal, and in number n .

$$2S = (a+l)n; \text{ then finally } S = \frac{(a+l)n}{2};$$

that is, the sum of the terms of a progression by difference is equal to the product of the sum of the extremes multiplied by half the number of terms.

If, in this formula, we substitute for l its value $a + (n-1)r$, we obtain further

$$S = \frac{[2a + (n-1)r]n}{2};$$

but the first expression is most used.

Applications. We require the sum of the first fifty terms of the progression,

$$2 . 9 . 16 . 23 . 30 . . .$$

We have, first, for the fiftieth term,

$$l = 2 + 49 . 7 = 345,$$

then

$$S = \frac{(2 + 345) . 50}{2} = 347 \times 25 = 8675.$$

We should find also, for the hundredth term,

$$l = 2 + 99 . 7 = 695;$$

and for the sum of the first hundred terms,

$$S = \frac{(2 + 695)100}{2} = 34850.$$

184. The formulas

$$l = a + (n - 1)r, S = \frac{(a + l)n}{2},$$

contain five quantities a , r , n , l , and S , and, consequently, give rise to this general problem; *Any three of these five quantities being given, to determine the other two.* This problem is subdivided into as many separate problems as we can, with the five letters, form different combinations, taken three and three or two and two. Now we have found (150) for the numbers of combinations, taken 2 and 2 and 3 and 3,

$$\frac{m(m-1)}{2} \text{ and } \frac{m(m-1)(m-2)}{2 \cdot 3}.$$

Making in these formulas $m = 5$, we find

$$\frac{5 \times 4}{2}, \text{ or } 10 \text{ and } \frac{5 \times 4 \times 3}{2 \cdot 3} \text{ or } 10;$$

whence we see that 5 letters combined 3 at a time, give the same number of combinations as 5 letters combined 2 at a time. (This result agrees with the consequence of article 153, by which the number of the combinations of m letters, taken p at a time, is equal to the number of combinations taken $m - p$ at a time.)

We see then that the above problem is divided into ten separate problems, of which the following are the enunciations:

- Having given
- 1st. a , r , n , to find l and S ;
 - 2d. a , r , l , ... n and S ;
 - 3d. a , r , S , ... n and l ;
 - 4th. a , n , l , ... r and S ;
 - 5th. a , n , S , ... r and l ;
 - 6th. a , l , S , ... r and n ;
 - 7th. r , n , l , ... a and S ;
 - 8th. r , n , S , ... a and l ;
 - 9th. r , l , S , ... a and n ;
 - 10th. n , l , S , ... a and r .

The first problem is already resolved, for the two formulas give immediately l and S in a function of a , r , n . As to the other problems, their resolution presents no difficulty; but we advise begin-

ners to perform them successively, this exercise being very well fitted to make them acquainted with the resolution of equations of the first and second degree; for it is worthy of remark, that although the quantities a , r , n , l , and S , enter only to the first degree in the two formulas, we are however led to the resolution of an equation of the second degree, when a and n , or rather l and n , are unknown, because a and n , or l and n , enter at the same time into the two equations, and are multiplied together in the second.

185. We will confine ourselves to the resolution of the fourth problem; it is the case in which, by knowing a , n , and l , it is required to determine r and S .

The formula

$$l = a + (n - 1)r \text{ gives } r = \frac{l - a}{n - 1},$$

and the formula

$$S = \frac{(a + l)n}{2}$$

shows immediately the value of S .

From the first expression,

$$r = \frac{l - a}{n - 1},$$

we deduce the solution of this question; to insert between two given numbers a and b , a number m of arithmetical means (we thus designate the numbers comprised between a and b , and forming with the latter a progression by difference.)

In order to resolve this last question, it is sufficient to determine the ratio; now, by substituting, in the above formulas, b for l , and $m + 2$ for n , which actually expresses the whole number of terms, we find

$$r = \frac{b - a}{m + 2 - 1}, \text{ or } r = \frac{b - a}{m + 1};$$

that is, the ratio of the required progression is obtained by dividing the difference of the two given numbers a and b by the number of terms to be inserted, plus one.

The ratio once obtained, we form the second term of the progression, or the first arithmetical mean, by adding r , or $\frac{b - a}{m + 1}$, to the first term a ; the second mean is obtained by increasing this by r , and so on.

Let it be required, for example, to insert 12 means between 12 and 77. We have

$$r = \frac{77 - 12}{13} = \frac{65}{13} = 5,$$

which gives the progression

$$\div 12 . 17 . 22 . 27 . 32 . 37 \dots 72, 77.$$

Consequence. *If between all the terms of a progression, considered two at a time, we insert an equal number of arithmetical means, these terms and the arithmetical means united form but one and the same progression.*

Finally, let $\div a . b . c . d . e . f . \dots$ be the proposed progression, and let m be the number of means which we wish to insert between a and b , b and c , c and $d \dots$

The ratio of each partial progression will be, according to what has just been said, expressed by

$$\frac{b-a}{m+1}, \frac{c-d}{m+1}, \frac{d-e}{m+1} \dots,$$

all equal quantities, since $a, b, c, \dots$ are in progression; thus the ratio is the same in each of the partial progressions, and as moreover the *last term* of the first forms the *first term* of the second, and so on, we may conclude that all these partial progressions constitute one uniform progression.

186. The following are the enunciations of several problems :

Question First. *To determine the first and the number of the terms of a progression by difference, of which the ratio is 6, the last term 185, and the sum 2945.*

(Ans. First term = 5, whole number of terms = 31.)

Second Question. *To insert between all the terms of the progression $\div 2 . 5 . 8 . 11 . 14 \dots$, nine arithmetical means.*

(Ans. Ratio or $r = 0,3$.)

Third Question. *To find the number of men contained in a triangular battalion of which the first rank is 1, the second 2, the third 3, and the ninth n . In other words, to find the expression of the sum of natural numbers 1, 2, 3 $\dots$ from 1 to n .*

$$\left(\text{Ans. } S = \frac{n(n+1)}{2} \right)$$

Fourth Question. *To find the sum of the n first terms of the progression of the uneven numbers, 1, 3, 5, 7, 9, $\dots$*

(Ans. $S = n^2$, or the square of the number of the terms.)

Fifth Question. *An avenue of trees is distant from a heap of sand 40 yards, and 100 loads are necessary to cover it, being placed at intervals of 6 yards from each other. Required the distance to be passed over by the team, the first load being deposited 40 yards from the heap, and the team returning at last to the place from which it started.* *Ans.* 67400 yards.

Sixth Question. *A foot soldier travels 10 leagues a day; a horseman sets out at the same time, and travels only 3 leagues the first day, but every succeeding day he travels 2 leagues more than on the preceding. It is required to find in how many days the horseman will overtake the foot soldier, and how much of the journey each will have performed.*

(Number of days, 8; distance, 80 leagues.)

Of Progression by Quotients.

187. We call *geometrical progression* or *progression by quotients* a series of terms of which each is equal to the product of that which precedes it, by a *constant number* which we call the *ratio* of the progression; thus the two series

$$3, 6, 12, 24, 48, 96 \dots,$$

$$64, 16, 4, 1, \frac{1}{4}, \frac{1}{16} \dots,$$

of which the first is such, that each term contains that which precedes it *twice*, or is equal to twice that which precedes it, and of which the second is such, that each term is contained in that which precedes it four times, or is equal to the fourth of that which precedes it, are called progressions by quotients; the ratio is 2 for the first, and $\frac{1}{4}$ for the second.

Let $a, b, c, d, e, f \dots$ be numbers in a progression by quotients; we write it thus,

$$\div a : b : c : d : e : f : g \dots,$$

and we enunciate it as a progression by difference, although there is this distinction to be made, that one is a series of equal differences, and the other a series of quotients or equal ratios in which each term is at once antecedent and consequent, except the first which is only an antecedent, and the last which is only a consequent.

188. Let us designate by q the ratio of the progression

$$\div a : b : c : d \dots,$$

q being > 1 , when the progression is *increasing*, and < 1 , if the progression is *decreasing*; we deduce from the same definition the series of equations

$$b = a q, c = b q = a q^2, d = c q = a q^3, e = d q = a q^4 \dots,$$

and in general, a term of any place whatever n , that is to say, which has $n - 1$ terms before it, has for its expression $a q^{n-1}$.

Let l be this term, we have the formula $l = a q^{n-1}$, by means of which we can obtain the value of any term whatever, without passing through all the terms which precede it. For example, the eighth term of the progression

$$\div 2 : 6 : 18 : 54 : \dots,$$

is equal to

$$2 \times 3^7 = 2 \times 2187 = 4374.$$

In a similar manner, the twelfth term of the progression

$$\div 64 : 16 : 4 : 1 : \frac{1}{4} \dots,$$

is equal to

$$64 \left(\frac{1}{4}\right)^{11} = \frac{4^3}{4^{11}} = \frac{1}{4^8} = \frac{1}{65536}.$$

189. Let it now be proposed to determine the sum of the n first terms of the progression

$$\div a : b : c : d : e : f \dots i : k : l,$$

l designating the n^{th} term,

We have (187) the equations

$$b = a q, c = b q, d = c q, e = d q \dots k = i q, l = k \times q;$$

whence we deduce, by adding them, member by member,

$$b + c + d + e + \dots + k + l = (a + b + c + d + \dots + i + k) q,$$

or rather, representing by S the sum required,

$$S - a = (S - l) q = S q - l q,$$

$$\text{or } S q - S = l q - a; \text{ then } S = \frac{l q - a}{q - 1};$$

that is to say, in order to obtain the sum of a determinate number of terms of a progression by quotients, we must multiply the last term by the ratio, subtract from the product the first term, and divide the difference by the ratio diminished by one.

When the progression is decreasing, we have $q < 1$, $l < a$, and it is proper to put the above formula under the form

$$S = \frac{a-lq}{1-q},$$

so that the two terms of the fraction may be positive.

The two expressions of S become moreover, by the substitution of $a q^{n-1}$ in the place of l ,

$$S = \frac{a q^n - a}{q - 1}, \text{ and } S = \frac{a^1 - a q^n}{1 - q}.$$

We shall find, according to the preceding formulas,

1. For the sum of the first 8 terms of the progression

$$\div 2 : 6 : 18 : 54 \dots : 2 \times 3^7 \text{ or } 4374,$$

$$S = \frac{lq - a}{q - 1} = \frac{13122 - 2}{2} = 6560.$$

2. For the sum of the first 12 terms of the progression

$$\div 64 : 16 : 4 : 1 : \frac{1}{4} : \dots : 64 \left(\frac{1}{4}\right)^{11} \text{ or } \frac{1}{65536},$$

$$S = \frac{a - lq}{1 - q} = \frac{64 - \frac{1}{65536} \cdot \frac{1}{4}}{1 - \frac{1}{4}} = \frac{256 - \frac{1}{16384}}{3} = 85 + \frac{65535}{196608}.$$

We see that the principal difficulty consists in determining the numerical value of the last term, a very laborious operation, when the number of terms is considerable.

190. *Remark.* If in the formula

$$S = \frac{a(q^n - 1)}{q - 1},$$

we suppose $q = 1$, it becomes $S = \frac{a}{q}$.

This result, which is sometimes the symbol of indeterminateness, proceeds also frequently (72) from the existence of a common factor, which becomes nothing, by a particular hypothesis made upon the given terms of the question. This is, indeed, what takes place in this instance; for we know (31) that the expression $q^n - 1$ is divisible by $q - 1$, and gives for a quotient

$$q^{n-1} + q^{n-2} + q^{n-3} + \dots + q + 1;$$

if we perform the division, the value of S takes the form

$$S = a q^{n-1} + a q^{n-2} + a q^{n-3} + \dots + a q + a;$$

whence, making $q = 1$,

$$S = a + a + a \dots + a = na.$$

We can arrive at the same result by going back to the proposed progression $\div a : b : c \dots l$, which, in this particular case of $q = 1$, is reduced to $\div a : a : a : a \dots a$, a series of which the sum is equal to $n a$.

The result $\frac{a}{2}$, which the formula gives, may further be regarded as showing an *insufficiency* of this formula, to give the expression of the sum, in this particular case. Indeed, the progression being composed of terms all equal among themselves, is not more a progression by quotient than a progression by difference. Thus, in seeking the sum of a certain number of terms, there is no more reason to make use of the formula

$$S = \frac{a(q^n - 1)}{q - 1},$$

than of the formula

$$S = \frac{(a + l)n}{2},$$

relating to progressions by difference.

191. *Of Infinite Progressions by Quotients.* Let there be a decreasing progression

$$\div a : b : c : d : e : f \dots$$

of an indefinite number of terms ; if we consider the formula

$$S = \frac{a - a q^n}{1 - q},$$

which gives the sum of a number n of terms, it may be put under the form

$$S = \frac{a}{1 - q} - \frac{a q^n}{1 - q}.$$

Now, since the progression is decreasing, q is a fraction ; q^n is also a fraction, which will be smaller in proportion as n will be larger ; so that the more terms we shall take in the progression, the more

$$\frac{a}{1 - q} \times q^n$$

will diminish ; consequently the more the partial sum of these terms will approach to being equal to the first part of S , that is, to

$\frac{a}{1 - q}$. Indeed, if we take for n a number greater than any

assignable quantity, or if we suppose $n = \infty$,

$$\frac{a}{1-q} \times q^n$$

will be less than any assignable quantity, or will become equal to 0; and the expression $\frac{a}{1-q}$ will represent the value of the whole series.

Whence we may conclude, that *the sum of the terms of a progression decreasing to infinity*, has for its expression

$$S = \frac{a}{1-q}.$$

This is, properly speaking, the *limit* towards which all the *partial sums* which we obtain are continually tending, by taking a greater and greater number of terms in the progression.

The difference between these sums and $\frac{a}{1-q}$, may become as small as we wish, and can never become entirely nothing, except when we take an infinite number of terms.

Applications. Let there be the progression decreasing to infinity

$$1 : \frac{1}{2} : \frac{1}{4} : \frac{1}{8} : \frac{1}{16} : \dots$$

We have, for the expression of the sum of the terms,

$$S = \frac{a}{1-q} = \frac{1}{1-\frac{1}{2}} = \frac{3}{2}.$$

The error which is committed by taking this expression for the value of the sum of the n first terms, is marked by

$$\frac{a}{1-q} q^n = \frac{3}{2} \cdot \left(\frac{1}{2}\right)^n.$$

Let there be at first $n = 5$; it becomes

$$\frac{3}{2} \left(\frac{1}{2}\right)^5 = \frac{1}{2 \cdot 3^4} = \frac{1}{162}.$$

For $n = 6$, we find

$$\frac{3}{2} \cdot \left(\frac{1}{2}\right)^6 = \frac{1}{162} \cdot \frac{1}{2} = \frac{1}{486}.$$

Whence we see that the *error committed*, when we take $\frac{3}{2}$ for the sum of a certain number of terms, is as much smaller as this number is greater.

Further, let there be the progression

$$1 : \frac{1}{2} : \frac{1}{4} : \frac{1}{8} : \frac{1}{16} : \frac{1}{32} : \dots$$

We have

$$S = \frac{a}{1-q} = \frac{1}{1-\frac{1}{2}} = 2.$$

The expression $S = \frac{a}{1-q}$ may be obtained directly according to the progression

$$\div a : b : c : d : e : f : g : \dots$$

Let us take again the equations

$$b = a q, c = b q, d = c q, e = d q \dots$$

of which the number here is indefinite, and add them member by member ; it becomes

$$b + c + d + e + \dots = (a + b + c + d + \dots) q.$$

Now the first member is evidently the proposed series, diminished by the first term a , and consequently it has for its expression $S - a$; the second member is equal to q multiplied by the whole series, since there is no last term, or that this last term is nothing ; the expression of this second member is then $q S$, and the above equation becomes

$$S - a = q S, \text{ whence we deduce } S = \frac{a}{1-q}.$$

Indeed, if we developpe $\frac{a}{1-q}$ into a series, by the process of division, we find for a result,

$$a + a q + a q^2 + a q^3 + \dots,$$

which is no other than the proposed series when we have substituted $b, c, d \dots$ for their values in a function of a .

Again, let there be the expression

$$\div a : a q : a q^2 : a q^3 \dots$$

and let us make

$$S = a + a q + a q^2 + a q^3 + a q^4 + a q^5 + \dots,$$

whence, multiplying the two members by q ,

$$q S = a q + a q^2 + a q^3 + a q^4 + a q^5 + \dots$$

Let us subtract the second of these two equations from the first ; we have

$$S - qS = a ; \text{ and lastly, } S = \frac{a}{1-q}.$$

192. When the series is increasing, the expression

$$S = \frac{a}{1-q}$$

can no longer be regarded as a limit of partial sums ; for the sum of a determinate number of terms being (199)

$$S = \frac{a}{1-q} - \frac{aq^n}{1-q},$$

the second part $\frac{aq^n}{1-q}$ increases more and more numerically in proportion as n increases ; that is, the more terms we take, the more the expression of the sum of these terms differs numerically from $\frac{a}{1-q}$.

The formula

$$S = \frac{a}{1-q}$$

is, in this case, simply the algebraic expression, which, by its development, gives the series

$$a + aq + aq^2 + aq^3 \dots$$

Here is presented a very singular circumstance. Since $\frac{a}{1-q}$ is the fraction which generates the series of which we have just spoken, we must have

$$\frac{a}{1-q} = a + aq + aq^2 + aq^3 + aq^4 + \dots$$

Now, by making in this equality $a = 1$, $q = 2$, we find

$$\frac{1}{1-2} \text{ or } -1 = 1 + 2 + 4 + 8 + 16 + 32 + \dots,$$

an equation of which the first number is negative, while the second is positive and so much the greater, according as q is greater.

To interpret this result, let us observe that when in the equation

$$\frac{a}{1-q} = a + aq + aq^2 + \dots,$$

we stop the series at a certain term, it is necessary in order that the equality should subsist to complete the quotient. Thus, in stopping,

for example, at the fourth term, $a q^3$,

$$\begin{array}{rcl}
 & a & | \quad 1 - q \\
 \text{1st remainder} & + a q & \\
 \text{2d} & + a q^2 & \\
 \text{3d} & + a q^3 & \\
 \text{4th} & + a q^4 & \\
 \hline
 & & a + a q + a q^2 + a q^3 + \frac{a q^4}{1 - q}
 \end{array}$$

we must add to the quotient obtained, the fractional expression

$\frac{a q^4}{1 - q}$, which gives rigorously

$$\frac{a}{1 - q} = a + a q + a q^2 + a q^3 + \frac{a q^4}{1 - q}.$$

If now we make, in this exact equation, $a = 1$, $q = 2$, it becomes

$$-1 = 1 + 2 + 4 + 8 + \frac{16}{-1} = 1 + 2 + 4 + 8 - 16;$$

an equality which is verified of itself.

In general, when an expression in terms of x , which we shall designate by $f(x)$, and which is called a function of x , is developed into a series of the form

$$a + b x + c x^2 + d x^3 + \dots,$$

we have rigorously

$$f(x) = a + b x + c x^2 + d x^3 + \dots,$$

only when on stopping at a certain term in the second member, we suppose the series completed by a certain expression in terms of x .

If in any particular application the series is *decreasing*, this expression, which serves to complete it, may be supposed as small as we please, when we prolong the series as far as it is convenient; but if the series is *increasing*, the contrary takes place, and this expression must not be neglected. For this reason, increasing series will not serve for obtaining the approximate value of numbers. It is for the same reason, that algebraists give the name *converging* to series whose terms go on diminishing, and *diverging* to series whose terms go on increasing. In the first, the more terms we take, the more the sum approaches numerically to the expression of which the series is the developement; while, on the contrary, in the others, the more terms we take, the more their sum differs from the numerical value of the expression reduced to a series.

193. *Remark.* We will conclude the principles relating to infinite progressions, by the following observation. It results from the definition of progression, by quotients (187), that we may regard them as recurrent series of the first order, of which the *scale of relation* is the *ratio* of the progression (art. 189). This approximation is fitted to make known the origin of progression prolonged to infinity. They owe, as well as recurrent series in general, their origin to the developement of an algebraic fraction into a series. We have given (190 and 191) the method for *finding this generating fraction for particular progressions*. We shall see hereafter the method for resolving the same question for all recurrent series.

194. The consideration of the five quantities a , q , n , l , and S , which enter into the two formulas

$$l = a q^{n-1}, S = \frac{lq - a}{q - 1},$$

obtained (arts. 188, 189) gives rise to ten particular problems of which the enunciations do not differ from the enunciations which relate to progression by difference, except that the letter q takes the place of r . But we propose, as in progressions by difference, to determine q and S , a , l , and n being known.

Now the first formula gives

$$q^{n-1} = \frac{l}{a}, \text{ whence } q = \sqrt[n-1]{\frac{l}{a}};$$

by carrying back this value into the second formula, we should obtain the value of S .

The expression

$$q = \sqrt[n-1]{\frac{l}{a}}$$

gives the method of resolving this question; *Insert between two given numbers a and b , a number m of mean proportionals, that is, a number m of quantities which form with a and b , considered as extremes, a progression by quotients.*

It is sufficient for this to know the *ratio*; now, the number of terms to be inserted being m , the whole number of terms n is equal to $m + 2$; we have moreover $l = b$; thus the value of q becomes

$$q = \sqrt[n+1]{\frac{b}{a}};$$

that is, it is necessary to divide the two given numbers b and a , the one by the other, then of the quotient to extract a root of a degree indicated by the number of the terms to be inserted plus unity.

The progression is then

$$\div a : a \sqrt[n+1]{\frac{b}{a}} : a \sqrt[n+1]{\frac{b^2}{a^2}} : a \sqrt[n+1]{\frac{b^3}{a^3}} : \dots b.$$

Hence let it be required to insert six mean proportionals between the numbers 3 and 384.

We have $m = 6$, whence

$$q = \sqrt[7]{\frac{384}{3}} = \sqrt[7]{128} = 2;$$

whence we deduce the progression

$$\div 3 : 6 : 12 : 24 : 48 : 96 : 192 : 384.$$

We shall soon show a more expeditious method in numerical applications, for calculating the number expressed by

$$q = \sqrt[n+1]{\frac{b}{a}}.$$

We shall not stop to demonstrate, that if between all the terms of a progression by quotients, considered two by two, we insert the same number of mean proportionals, all the progressions thus formed, constitute a unique progression. The demonstration is analogous to that of 185.

195. Of the ten principal problems which may be proposed concerning progressions, four are capable of being easily resolved.

The following are the enunciations, with the formulas which relate to them :

1. a, q, n , being given, to find l and S .

$$l = a q^{n-1}; S = \frac{l q - a}{q - 1} = \frac{a (q^n - 1)}{q - 1}.$$

2. a, n, l , being given, to find q and S .

$$q = \sqrt[n-1]{\frac{l}{a}}; S = \frac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}.$$

3. q, n, l , being given, to find a and S .

$$a = \frac{l}{q^{n-1}}, \quad S = \frac{l(q^n - 1)}{q^{n-1}(q - 1)}.$$

4. q, n, S , being given, to find a and l .

$$a = \frac{S(q - 1)}{q^n - 1}; \quad l = \frac{S q^{n-1}(q - 1)}{q^n - 1}.$$

Two other problems depend on the resolution of equations of a degree higher than the second; they are those in which we suppose the quantities a and q , or rather l and q unknown.

Indeed from the second formula we deduce

$$a = lq - Sq + S;$$

whence, substituting this value of a in the first formula $l = a q^{n-1}$,

$$l = (lq - Sq + S) q^{n-1},$$

or rather

$$(S - l) q^n - S q^{n-1} + l = 0,$$

an equation of the degree n which we have not yet learned to resolve.

The same will take place, if we wish to determine l and q ; we shall arrive at the equation

$$a q^n - Sq + S - a = 0.$$

196. Finally, the *four* other problems lead to the resolution of equations of a very peculiar nature; they are those in which n is the unknown quantity, as well as one of the four other quantities.

At first, the second formula easily gives the value of one of the quantities a, q, l , and S , in a *function* of the three others; so that the whole is reduced to determining n by means of the formula

$$l = a q^{n-1}.$$

Now this equality is reduced to $q^n = \frac{lq}{a}$, an equation of the form $a^x = b$, a and b being known quantities. We call these equations *exponential equations*, to distinguish them from those which we have hitherto considered, in which the unknown quantity is raised to powers indicated by known numbers.

Let us now attend to the resolution of these equations, to which is attached one of the most important theories in the mathematics, the theory of logarithms.

II. Of the Theory of Exponential Quantities and Logarithms.

197. *Resolution of the Equation* $a^x = b$. The question consists in finding the exponent of the power to which we must raise a given number a , to produce another given number b .

Let us at first consider some particular cases.

Let it be required to resolve the equation $2^x = 64$. By raising 2 to its different powers, we soon discover that $2^6 = 64$; then $x = 6$ satisfies the equation.

Further, let there be the equation $3^x = 243$. We have for a solution $x = 5$. In short, so long as the second member b shall be a *perfect power* of the given number a , x will be a whole number which we shall obtain by raising a to its successive powers, beginning from the degree 0.

Let it now be required to resolve the equation $2^x = 6$. By making $x = 2$ and $x = 3$, we find $2^2 = 4$ and $2^3 = 8$; whence we see that x has a value comprehended between 2 and 3.

Let us put then

$$x = 2 + \frac{1}{x'} \dots (x' \text{ is then } > 1).$$

We have, by substituting this value in the proposed equation,

$$2^{2+\frac{1}{x'}} = 6, \text{ or (art. 180) } 2^2 \times 2^{\frac{1}{x'}} = 6; \text{ then } 2^{\frac{1}{x'}} = \frac{3}{2}.$$

or, raising the two members to the power x' , $\left(\frac{3}{2}\right)^{x'} = 2$.

In order to determine x' , let us make successively

$$x' = 1, x' = 2; \text{ we find } \left(\frac{3}{2}\right)^1 = \frac{3}{2},$$

a number smaller than 2, and $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$, a number greater than 2; hence x' is comprehended between 1 and 2.

Let us put then

$$x' = 1 + \frac{1}{x''} \dots (x'' \text{ is also } > 1).$$

We obtain, by substituting in the exponential equation of x' ,

Bour. Alg.

$$\left(\frac{3}{2}\right)^{1+\frac{1}{x''}} = 2, \text{ or } \frac{3}{2} \times \left(\frac{3}{2}\right)^{\frac{1}{x''}} = 2,$$

or reducing

$$\left(\frac{4}{3}\right)^{x''} = \frac{3}{2}.$$

The two hypotheses

$$x'' = 1 \text{ and } x'' = 2, \text{ give } \left(\frac{4}{3}\right)^1 = \frac{4}{3},$$

a number smaller than $\frac{3}{2}$, and

$$\left(\frac{4}{3}\right)^2 = \frac{16}{9} = 1 + \frac{7}{9},$$

a number greater than $\frac{3}{2}$; hence x'' is comprehended between 1 and 2.

Let then $x'' = 1 + \frac{1}{x'''};$ there results

$$\left(\frac{4}{3}\right)^{1+\frac{1}{x'''}} = \frac{3}{2}, \text{ or } \frac{4}{3} \times \left(\frac{4}{3}\right)^{\frac{1}{x'''}} = \frac{3}{2};$$

whence reducing,

$$\left(\frac{9}{8}\right)^{x'''} = \frac{4}{3}.$$

If we make successively $x''' = 1, 2, 3$, we find, for the two last hypotheses,

$$\left(\frac{9}{8}\right)^1 = \frac{81}{64} = 1 + \frac{17}{64},$$

a number $< 1 + \frac{1}{3}$, and

$$\left(\frac{9}{8}\right)^3 = \frac{729}{512} = 1 + \frac{217}{512},$$

a number $> 1 + \frac{1}{3}$; so that x''' is comprehended between 2 and 3.

Let $x''' = 2 + \frac{1}{x^{iv}}$, the equation in x''' becomes

$$\left(\frac{9}{8}\right)^{2+\frac{1}{x^{iv}}} = \frac{4}{3}, \text{ or } \frac{81}{64} \left(\frac{9}{8}\right)^{\frac{1}{x^{iv}}} = \frac{4}{3};$$

and, consequently,

$$\left(\frac{256}{243}\right)^{x^{iv}} = \frac{9}{8}.$$

By proceeding in this exponential equation as in the preceding, we should find two whole numbers, k and $k + 1$, between which x^v would be comprehended. Putting

$$x^v = k + \frac{1}{x^v},$$

we should determine x^v as we have determined x^v ; and so on.

Let us bring together the equations

$$x = 2 + \frac{1}{x}, x' = 1 + \frac{1}{x'}, x'' = 1 + \frac{1}{x''}, x''' = 2 + \frac{1}{x^v} \dots;$$

we obtain the value of x under the form of a continued fraction

$$x = 2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{x^v}}}}$$

Now, we know (Arith. art. 175), that in a continued fraction the greater number of integral fractions we take, the nearer we approach to the value of the number reduced to a continued fraction; so that we can, by these means, find the value of x fitted to verify the equation $2^x = 6$, if not exactly, at least to any degree of approximation that we wish.

For example, by forming the first four results according to the established law (Arith. art. 169), we find

$$\frac{2}{1}, \frac{3}{1}, \frac{5}{2}, \frac{1}{2},$$

and the result $\frac{1}{2}$ differs (Arith. 174) from the value of x only by a quantity less than $\frac{1}{(5)^2}$ or $\frac{1}{25}$. But the approximation is still greater; for if we calculate the value of x^v according to the equation

$$\left(\frac{256}{243}\right)x^v = \frac{9}{8},$$

we shall discover that x^v is comprehended between 2 and 3; so that

$$x^v = 2 + \frac{1}{x^v};$$

then the 5th result is

$$\frac{13 \times 2 + 5}{5 \times 2 + 2}, \text{ or } \frac{31}{12}.$$

So that $\frac{1}{5}$ differs from the value of x , by a quantity less than

$$\frac{1}{12 \times 5}, \text{ or } \frac{1}{60}.$$

The result $\frac{2}{11}$ differs from it less than

$$\frac{1}{(12)^2}, \text{ or } \frac{1}{144}.$$

General Method. Let $a^x = b$ be the equation to be resolved. By forming the successive powers of a , we find that b is comprehended between a^n and a^{n+1} ; then we make

$$x = n + \frac{1}{x'}.$$

Substituting this value in the equation, we obtain

$$a^{n+\frac{1}{x'}} = b,$$

an equation which returns to

$$a^n \times a^{\frac{1}{x'}} = b, \text{ whence } \left(\frac{b}{a^n}\right)^{x'} = a;$$

or putting, for the sake of greater simplicity,

$$\frac{b}{a^n} = c \dots c^{x'} = a.$$

Proceeding with the operation in this equation as in the proposed equation, we shall discover that x' is comprehended between n' and $n' + 1$, which will give

$$x' = n' + \frac{1}{x''};$$

substituting this value in the equation of x' , we shall further be led to resolve an equation of the form $d^{x''} = c$ (d having for its value $\frac{a}{c^{n'}}$), and so on. Then, finally, we shall obtain for the value of x an expression of the form

$$x = n + \frac{1}{n'} + \frac{1}{n''} + \frac{1}{n'''} + \dots$$

By carrying on the series of operations to a convenient length, we shall have the value of x with any degree of approximation that we

wish ; and this degree can always be estimated, since it is indicated (Arith. art. 174) *by the quotient of one divided by the square of the denominator of the last result at which we have arrived.*

198. *Remark 1.* If we suppose, in the equation $a^x = b$, $b < a$, as we have $a^0 = 1$ (24) and $a^1 = a$, it follows that x is comprehended between 0 and 1, and we must then put $x = \frac{1}{x'}$.

2. If b is a fraction and a is greater than unity, we must put in the equation

$$a^x = b, \quad x = -y,$$

which gives $a^{-y} = b$, whence (179)

$$a^y = \frac{1}{b}; \text{ and as } \frac{1}{b} \text{ is } > 1,$$

we shall determine y , according to the foregoing method, and the corresponding value of x will be equal to that of y taken *negatively*.

By means of these remarks, the application of the method offers no difficulty ; only the calculations, in order to obtain a great degree of approximation, are very laborious.

We can, for the remainder, perform the following examples as an exercise.

$$\begin{aligned} 3^x &= 15 \dots x = 2,46 \text{ to } 0,01 \text{ nearly,} \\ 10^x &= 3 \dots x = 0,477 \text{ to } 0,001 \text{ nearly,} \\ 5^x &= \frac{2}{3} \dots x = -0,25 \text{ to } 0,01 \text{ nearly,} \\ \left(\frac{7}{12}\right)^x &= \frac{3}{4} \dots x = 0,53 \text{ to } 0,01 \text{ nearly.} \end{aligned}$$

We suppose here that we have converted to decimal fractions, the results furnished by the method.

199. It may be asked, whether by following the preceding method we shall be led to a continued fraction of a limited number of integral fractions, which will give for x a *commensurable* number and equal to the last result of the continued fraction ; or indeed, whether the number of integral fractions will be unlimited, in which case x will be *incommensurable*.

In order to answer this question, let us suppose in the equation $a^x = b$, x equal to a commensurable number $\frac{m}{n}$, and let us see what relation will exist between the numbers a and b , in order that this value may be admitted, that is, in order that x may be *commensurable*.

In the first place, let a and b be two whole numbers; we have the equation $a^{\frac{m}{n}} = b$, which we may put under the form $a^m = b^n$.

It is at once evident that this equality can only subsist when a and b are composed of the same prime factors; for if we suppose in b a prime factor which is not found in a , and divide the two members by this factor, the second member will be a whole number, and the first a fractional number, which is absurd; then, if we have, for example,

$$a = \alpha^p 6^q \gamma^r \delta^s,$$

we must also have

$$b = \alpha^{p'} 6^{q'} \gamma^{r'} \delta^{s'}.$$

Substituting these values in the equation $a^m = b^n$, we change it to this,

$$\alpha^{mp} 6^{mq} \gamma^{mr} \delta^{ms} = \alpha^{np'} 6^{nq'} \gamma^{nr'} \delta^{ns'}.$$

This new equality can evidently subsist only as long as the powers of the same prime factor are equal in the two members; for, if they were unequal, by dividing the two members by the highest power, we should still be led to this absurd result; viz. *a whole number equal to a fractional number*.

Thus, we ought to have separately

$$mp = np', \quad mq = nq', \quad mr = nr', \quad ms = ns',$$

whence we deduce

$$\frac{m}{n} = \frac{p'}{p} = \frac{q'}{q} = \frac{r'}{r} = \frac{s'}{s}.$$

Then in order that the value of x may be commensurable, it is necessary and sufficient that a and b be composed of the same prime factors, and that the exponents of these factors form between themselves a series of equal ratios. If these two conditions are satisfied, the value of x is equal to the constant ratio which exists between the exponents.

Let us suppose, in the second place, that a and b are fractional and equal to $\frac{h}{h'}$, $\frac{k}{k'}$; the equation $a^m = b^n$ becomes

$$\left(\frac{h}{h'}\right)^m = \left(\frac{k}{k'}\right)^n, \text{ whence } h^m k^n = h'^m k'^n.$$

Now, h and h' , k and k' being always considered as prime among themselves, it is the same with h^m and h'^m , k^n and k'^n ; so

that, in order that the preceding equality may exist, we must have separately

$$h^m = k^n, h'^m = k'^n,$$

which leads to conditions like those above, between the numerators and denominators respectively compared.

Examples. 1. If a and b , being whole numbers, contain but one, single prime factor, x is necessarily commensurable.

Let there be the equation $4^x = 32$, which becomes $2^{2x} = 2^5$; there results $2x = 5$, whence $x = \frac{5}{2}$.

Further, let there be $27^x = 2187$, or $3^{3x} = 3^7$; we have $x = \frac{7}{3}$.

2. If a is composed of prime factors only, raised to the first power, it is necessary that b should be a perfect power of a , in order that x may be commensurable; thus x is, in this case, a whole number, or rather, it is incommensurable.

Finally, let

$$a = \alpha \gamma \delta, \text{ whence } b = \alpha^{p'} \gamma^{q'} \delta^{r'};$$

the equation $a^m = b^n$ becomes

$$\alpha^m \gamma^m \delta^m = \alpha^{p'n} \gamma^{q'n} \delta^{r'n};$$

whence we deduce

$$m = p'n = q'n = r'n;$$

or rather

$$p' = q' = r' = s';$$

then

$$b = \alpha^{p'} \gamma^{p'} \delta^{p'} = (\alpha \gamma \delta)^{p'} = a^{p'},$$

and, consequently $x = p'$.

Thus, let $a = 10 = 2 \times 5$; it is necessary that b should be a perfect power of 10, in order that x may be commensurable.

Theory of Logarithms.

200. *Introduction.* If we suppose that in the equation $a^x = y$, a preserving always the same value, we replace y by all possible absolute numbers, we can, for each value of y , determine by the method of (198) the value of x , if not exactly, at least to as great a degree of approximation as we wish.

Let us suppose at first $a > 1$.

If we make successively

$$x = 0, 1, 2, 3, 4, 5 \dots,$$

there results

$$y = a^0, \text{ or } 1, a, a^2, a^3, a^4, a^5 \dots;$$

then, *all the values of y greater than unity are produced by powers of a, of which the exponents are positive, whole or fractional; and the value of x is so much the greater as that of y is greater.*

Let us make then,

$$x = 0, -1, -2, -3, -4, -5 \dots,$$

there results

$$y = a^0, \text{ or } 1, \frac{1}{a}, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4}, \frac{1}{a^5} \dots;$$

then, *all the values of y less than unity are produced by powers of a, of which the exponents are negative; and the value of x is so much the greater, negatively, as the value of y approaches nearer to zero.*

On the contrary, let there be $a < 1$ and equal to a fraction $\frac{1}{a'}$; by making

$$x = 0, 1, 2, 3, 4, 5 \dots,$$

we find $y = \left(\frac{1}{a'}\right)^0,$

$$\text{or } 1, \frac{1}{a'}, \frac{1}{a'^2}, \frac{1}{a'^3}, \frac{1}{a'^4}, \frac{1}{a'^5} \dots,$$

and if we make

$$x = 0, -1, -2, -3, -4, -5 \dots,$$

we obtain $y = \left(\frac{1}{a'}\right)^0,$

$$\text{or } 1, a', a'^2, a'^3, a'^4, a'^5 \dots;$$

that is, on the hypothesis of $a < 1$, all the numbers are produced with the different powers of a , in an inverse order from that in which they are, when we suppose $a > 1$.

But there results no less this general consequence, that *all possible absolute numbers may be produced with any absolute, but invariable number whatever, by raising it to suitable powers.*

. N. B. We must always suppose *a different from unity*, for we know that all the powers of 1 are equal to 1.

201. This being laid down, let us suppose that we have formed a table containing, on one hand, all whole numbers, and on a line with these numbers, *the exponents of the powers to which we must raise an invariable number* in order to form all these numbers; we shall then have an idea of a table of *logarithms*.

We generally call *the logarithm of a number the exponent of the power to which it is necessary to raise a certain invariable number, in order to produce the first number*.

The invariable number may at first be taken altogether arbitrarily, (provided that it be $>$ or $<$ 1); but when once chosen it must remain the same for the formation of all numbers, and it is called the *base* of the system of logarithms.

Whatever may be the base which we may have chosen, *the logarithm of the base is unity, and the logarithm of 1 is zero*.

For we have,

$$1. a^1 = a, \text{ whence } \log a = 1;$$

$$2. a^0 = 1, \text{ whence } \log 1 = 0.$$

(We generally designate, for the sake of brevity, the word logarithm by the three first letters *log*, or simply by the first letter *l*, after which we place a *period*, and then the number which we are to consider.)

Let us see what properties actually belong to a table of logarithms, with reference to numerical calculations.

202. *Arithmetical Multiplication and Division*. Let there be, in the first place, a series of numbers, $y, y', y'', y''' \dots$ to be multiplied with each other. Let us designate by a the base of a system of logarithms (which we suppose already calculated); and by $x, x', x'', x''' \dots$ the logarithms of $y, y', y'', y''' \dots$

We have, according to the definition (208), the series of equations

$$y = a^x, y' = a^{x'}, y'' = a^{x''}, y''' = a^{x'''} \dots$$

Multiplying these equations, member by member, and applying the rule of the exponents (180), we find

$$y y' y'' y''' \dots = a^{x+x'+x''+x'''+\dots}$$

Therefore,

$\log y y' y'' \dots = x + x' + x'' + \dots = \log y + \log y' + \log y'' + \dots$; that is, *the logarithm of a product is equal to the sum of the logarithms of the factors of that product*.

Bour. Alg.

Let there be, in the second place, two numbers, y and y' , to be divided the one by the other, x and x' being their logarithms; we have the equations

$$y = a^x, y' = a^{x'};$$

whence we deduce (180)

$$\frac{y}{y'} = a^{x-x'}.$$

Therefore,

$$\log \frac{y}{y'} = x - x' = \log y - \log y';$$

that is, *the logarithm of the quotient of a division is equal to the difference between the logarithm of the dividend and the logarithm of the divisor.*

Consequences of these two properties. If we have a multiplication to perform, by taking in the table the logarithms of the factors, and adding together these logarithms, we shall have the logarithm of the product; then, by seeking this new logarithm in the table, and taking the number which corresponds to it, we shall obtain the product required. So that *by a simple addition we find the result of a multiplication.*

In like manner, if we wish to divide one number by another, we subtract the logarithm of the divisor from that of the dividend, then we find what number corresponds to the difference; this is the required quotient. Thus, *by a simple subtraction, we obtain the quotient of a division.*

203. Formation of Powers and extraction of Roots. Let there be, in general, a number y to be raised to the power $\frac{m}{n}$; designating by a the base, and by x the logarithm of y , we have the equation

$$y = a^x,$$

whence, by raising the two members to the power $\frac{m}{n}$,

$$y^{\frac{m}{n}} = a^{\frac{m}{n} \cdot x}.$$

Therefore,

$$\log y^{\frac{m}{n}} = \frac{m}{n} \cdot x = \frac{m}{n} \cdot \log y;$$

that is, *the logarithm of any power whatever of a number is equal*

to the product of the logarithm of the number by the exponent of the power.

Let there be, as an example, $n = 1$; there results

$$\log y^m = m \cdot \log y,$$

an equation susceptible of an enunciation analogous to the preceding.

Let there now be $m = 1$, n being any number whatever; there results

$$\log y^{\frac{1}{n}} \text{ or } \log \sqrt[n]{y} = \frac{1}{n} \cdot \log y;$$

that is, *the logarithm of a root, of whatever degree, of a number is equal to the quotient arising from the division of the logarithm of that number by the index of the root.*

Consequence. In order to form any power of a number, it is sufficient to take the logarithm of that number in the table, and multiply it by the exponent of the power, then to seek the number corresponding to that product. We have thus the power required.

In the same manner, in order to extract a root, it is sufficient to divide the logarithm of the proposed number by the index of the root, then to find the number corresponding to the quotient, and we have the root required. Thus, *by a simple multiplication and a simple division we find the result of the formation of a power and of the extraction of a root*, operations in which, as we have seen, the process is very laborious.

204. The properties which we have just demonstrated are independent of every system of logarithms; but the consequences which have been deduced from them, that is, the use which we can make of them in numerical calculations, suppose the construction of a table, containing on one side all the numbers, and on the other the *logarithms* of those numbers, calculated according to a given *base*. Now in order to form this table, we must, as we have already said, in considering the equation $a^x = y$, give to y all possible values, and determine the value of x , corresponding to each of the values of y , according to the method of (297).

The tables which are generally used, are those whose *base* is equal to 10, and their construction is reduced to resolving the equation $10^x = y$. By successively making y equal to the series of natural numbers

$$1, 2, 3, 4, 5, 6 \dots,$$

we have the equations

$$10^x = 1, 10^x = 2, 10^x = 3, 10^x = 4 \dots$$

Let us observe, moreover, that it is sufficient to calculate directly, according to the method of article 297, the logarithms of the prime numbers

$$1, 2, 3, 5, 7, 11, 13, 17 \dots;$$

for all the other whole numbers, resulting from the multiplication of these different factors, their logarithms may (213) be obtained by the addition of the logarithms of the prime numbers.

It is thus that 6 being decomposable into 2×3 , we have

$$\log 6 = \log 2 + \log 3;$$

in the same way

$$24 = 2^3 \times 3; \text{ then } \log 24 = 3 \log 2 + \log 3.$$

Again, let there be

$$360 = 2^3 \times 3^2 \times 5;$$

there results

$$\log 360 = 3 \log 2 + 2 \log 3 + \log 5.$$

It would be equally sufficient to place in the tables the logarithms of whole numbers; for by means of the property (203), relating to division, we obtain the logarithm of a fractional number, by subtracting the logarithm of the divisor from that of the dividend.

205. By supposing a first table of logarithms already constructed, it is easy to construct as many others as we please after the method of the former.

For, let a be the base of a first system already formed, b the base of a new system to be constructed; let us designate by N any number whatever, by $\log N$ and X , its two logarithms, calculated according to the bases a and b ; we have the equation

$$b^X = N.$$

Whence by taking the logarithms of the two members in the system of which the base is a ;

$$X \log b = \log N.$$

Therefore

$$X = \frac{\log N}{\log b};$$

which proves that, by knowing the logarithm of a number in a first system, in order to have the logarithm of the same number in a second system, we must divide the logarithm of the number, calcu-

lated in the first system, by the logarithm of the new base, also calculated in the former system.

Hence the logarithm of 4, in the system of which the base is 3, has for its value $\frac{\log 4}{\log 3}$, $\log 4$ and $\log 3$ being two logarithms calculated in the known system, whose base is 10.

Let $N, N', N'' \dots$ be a series of numbers, a the base of a system already formed, b that of a system to be constructed; we have the series of equations,

$$X = \frac{\log N}{\log b} = \frac{1}{\log b} \cdot \log N; \quad X' = \frac{1}{\log b} \cdot \log N';$$

$$X'' = \frac{1}{\log b} \cdot \log N'' \dots;$$

whence we see, that a first table being already formed, if we wish to construct a new one, we have only to multiply the logarithms of the first system by the constant quantity $\frac{1}{\log b}$. This constant quantity, which enables us to pass from one table to another, is called the *modulus* of the new table with reference to the old table.

III. Application of the Theory of Logarithms.

206. Multiplication and Division. Let it be required to find the approximate value of the product $\frac{31}{7} \times \frac{13}{12} \times \frac{47}{48}$.

Calling this product x , we have (215),

$$\log x = \log 31 - \log 75 + \log 13 - \log 12 + \log 47 - \log 48.$$

$$\log 31 = 1,49136169$$

$$\log 13 = 1,11394335,$$

$$\log 47 = 1,67209786,$$

$$\text{arith. } \log 75 = 8,12493874,$$

$$\text{arith. } \log 12 = 8,92081875,$$

$$\text{arith. } \log 48 = 8,31875876,$$

$$\begin{array}{r} - \\ 1,64191915 = 29,64191915 - 30; \end{array}$$

$$\text{adding} \quad . \quad . \quad . \quad 5$$

$$\text{we obtain} \quad . \quad . \quad . \quad 4,6419191$$

$$4,6419120 = \log 43844.$$

$$\begin{array}{r} \text{Difference} \quad . \quad . \quad . \quad 89 \\ \text{Tabular difference} \quad . \quad . \quad . \quad 99 \end{array} \left| \begin{array}{l} 89 \\ 99 \end{array} \right. = 0,90.$$

$$\text{Then} \quad . \quad . \quad . \quad . \quad 4,6419191 = \log 43844,90;$$

the product required is 0,4384490 to within 0 0000001 nearly.

Formation of the Powers. We observe, in the first place, that since in order to obtain the result of raising a number to any power, we must multiply the logarithm of the number by the exponent of the power, we must take at first the logarithm of the proposed number with more than 7 decimals, if we wish to have a product exact to the 7th decimal inclusive. Now, we find in the work of Callet, at the end of the common tables, another table, which gives logarithms with 20 decimals; so that we can always take these logarithms with two or three decimals more than in the common tables.

This being premised, let it be required to form the 5th power of 29; we have (216)

$$\begin{array}{lcl} & \log (29)^5 = 5 \log 29 ; \\ \text{now} & \log 29 = 1,462397998, \\ \text{whence} & 5 \log 29 = 7,311989990 ; \\ \text{subtracting 3 units} & 4,3119900 \\ & 4,3119868 = \log 20511. \end{array}$$

$$\begin{array}{lcl} \text{Difference} & 32 & | \quad 32 \\ \text{Tabular difference} & 212 & | \quad 212 = 0,15 ; \end{array}$$

then 20511150 is the number sought to within a *tenth*.

Let it further be proposed to find the value of $(2)^{64}$.

$$\begin{array}{lcl} \text{We have} & \log 2 = 0,3010299956 ; \\ \text{whence} & 64 \log 2 = 19,2659197184. \\ \text{Subtracting 15 units,} & 4,2659197 \\ & 4,2659022 = \log 18446. \end{array}$$

$$\begin{array}{lcl} \text{Difference} & 175 & | \quad 175 \\ \text{Tabular difference} & 235 & | \quad 235 = 0,74. \end{array}$$

$$\text{So that} \quad 4,2659197 = \log 18446,74.$$

Then the number sought is 18.446.740.000.000.000.000, to within *tens of trillions*; that is, the thirteen last figures cannot be given by the tables; but our only object in examples of this kind is, to form an idea of the greatness of the number; and it is seen with what readiness it is obtained.

Let us seek, for a new example, the value of $\left(\frac{2}{3}\right)^{11}$.

The following are the calculations; by making use of the **Arithmetical complements**, and without them.

By Arithmetical complements.

$$\begin{array}{rcl} \log 2 & = & 0,3010299956 \\ \text{c. } \log 3 & = & 9,5228787453 \\ & & \hline \log \frac{2}{3} & = & -1,8239087409 \\ 11 \log \frac{2}{3} & = & -2,0629961499 \\ \text{adding} & 6 & \hline \end{array}$$

we obtain 4,0629961.

The number corresponding to this logarithm is 11561,02; then 0,01156102 is the number required to within 0,00000001.

Without the Arithmetical complements.

$$\begin{array}{rcl} \log 3 & = & 0,4771212547 \\ \log 2 & = & 0,3010299956 \\ & & \hline \log \frac{2}{3} & = & -0,1760912591 \\ 11 \log \frac{2}{3} & = & -1,9370038510 \\ \text{adding} & + 6 & \hline \end{array}$$

we obtain 4,0629961.

The remainder of the calculation is the same as that on the other side.

Extraction of the Roots. It is sufficient, in this operation, to take the logarithms with seven decimals.

We require the 7th root of 1162049.

We have (216),

$$\log \sqrt[7]{1162049} = \frac{1}{7} \log 1162049.$$

	$\log 11620 = 0652061$	Tabular difference	374
$\log 11620,49 - \log 11620 =$	183	Diff. of the num.	0,49
	$\log 1162049 = 0652244$		3366
Then $\log 1162049 =$	6,0652244		1496
$\frac{1}{7} \log 1162049 =$	0,8664606		183,26
adding 4,	4,8664606		
	$4,8664587 = \log 73529.$		

$$\begin{array}{rcl} \text{Difference} & 19 \overline{) 19} & \\ \text{Tabular difference} & 59 \overline{) 59} & = 0,32; \end{array}$$

then $4,8664606 = \log 73529,32.$

So that 7,352932 is the root required to within 0,000001.

Let it be required to find the value of

$$\sqrt[11]{\frac{13}{27}}; \text{ we have } \log \sqrt[11]{\frac{13}{27}} = \frac{1}{11} (\log 13 - \log 27).$$

By Arithmetical complements.

$$\begin{array}{rcl} \text{l. } 13 & = & 1,11394335 \\ \text{compl. l. } 27 & = & 8,56863624 \\ & & \hline \text{l. } \frac{13}{27} & = & -1,68257959 = -11 + 10,68257959 \\ \frac{1}{11} \text{ l. } \frac{13}{27} & = & -1,97114360; \\ \text{adding} & 5 & \hline \text{we find} & & 4,97114360 = \log 93571,49. \end{array}$$

Then the root required is 0,9357149, within 0,0000001.

We shall find, in the same manner,

$$\sqrt[7]{\left(\frac{11}{9}\right)^5} = 1,154118; (73)^7 = 11047390000000;$$

$$(0,0457)^{12} = 0,00000000000000082984.$$

207. *Calculation of Algebraic Expressions by Logarithms.*

Let us suppose that we have found, for the value of the unknown quantity of a problem, the expression

$$x = \frac{\sqrt[3]{(a^2 - b^2) \cdot 3a}}{\sqrt{(a+b) \cdot \sqrt{cd}}},$$

and that by giving to a, b, c, d , particular values, we wish to obtain the numerical value corresponding to this expression; we can, by means of logarithms, reduce the question so as to have only additions and subtractions, simple multiplications and divisions to perform.

We have, in short, according to the properties (215 and 216).

$$l. x = l. \sqrt[3]{(a^2 - b^2) \cdot 3a} - l. \sqrt{(a+b) \sqrt{cd}}.$$

But

$$l. \sqrt[3]{(a^2 - b^2) \cdot 3a} = \frac{1}{3} [l. (a+b) + l. (a-b) + l. 3 + l. a]$$

and

$$l. \sqrt{(a+b) \sqrt{cd}} = \frac{1}{2} [l. (a+b) + \frac{1}{2} l. c + \frac{1}{2} l. d];$$

then

$$l. x = \frac{1}{3} [l. (a+b) + l. (a-b) + l. 3 + l. a] \\ - \frac{1}{2} [l. (a+b) + \frac{1}{2} l. c + \frac{1}{2} l. d],$$

an expression which only requires additions, subtractions, and simple divisions to perform, when a, b, c, d , are given numerically.

For example, let

$$a = 60, b = 15, c = 16, d = 9;$$

the expression becomes

$$l. x = \frac{1}{3} [l. 75 + l. 45 + l. 3 + l. 60] - \frac{1}{2} [l. 75 + \frac{1}{2} l. 16 + \frac{1}{2} l. 9];$$

calculating separately the sum which is between the two first parentheses, and the sum which is between the other two, then taking a *third* of the first, and *half* of the second, we shall find

l. $x = 1,92784875 - 1,47712125$,
 or l. $x = 0,4507275$;
 then $x = 2,823108$.

Let there be further the expression

$$x = \frac{2a^4 - 3ab^3 + b^4}{a^3 - 3a^2b + 4b^2c}$$

We can at first, by separating the factor a^3 in the numerator, and the factor a^3 in the denominator, present the expression under the form

$$x = \frac{a^3 \left(2a - \frac{3b^3}{a^2} + \frac{b^4}{a^3} \right)}{a^3 \left(a - 3b + \frac{4b^2c}{a^2} \right)} = \frac{a \left(2a - \frac{3b^3}{a^2} + \frac{b^4}{a^3} \right)}{a - 3b + \frac{4b^2c}{a^2}}$$

Let us now put

$$m = \frac{4b^2c}{a^2}, \quad n = \frac{3b^3}{a^2}, \quad p = \frac{b^4}{a^3};$$

the expression becomes

$$x = \frac{a(2a - n + p)}{a - 3b + m};$$

or applying the logarithms,

$$l. x = l. a + l. (2a - n + p) - l. (a - 3b + m),$$

an expression easily calculated, as soon as we have found the values of m, n, p . Now the equations

$$m = \frac{4b^2c}{a^2}, \quad n = \frac{3b^3}{a^2}, \quad p = \frac{b^4}{a^3},$$

give

$$l. m = l. 4 + 2l. b + l. c - 2l. a, \quad l. n = l. 3 + 3l. b - 2l. a, \\ l. p = 4l. b - 3l. a.$$

The art of these transformations consists in reducing the fractional expression to another, the terms of which are all of the first degree, by calculating separately the other expressions which require only multiplications, divisions, and formations of powers.

We shall find, in the same manner,

$$l. \frac{a^2 - b^2}{b d} = l. (a + b) + l. (a - b) + \text{comp. } l. b + \text{comp. } l. d - 20,$$

$$1. \frac{a^2 - 2ba^2 + bc^2}{a^2 - ba + 4ca} = 1. a + 1. (a - 2b + h) + \text{comp.} 1. (a - b + h) - 10,$$

h and h' being calculated according to the formulas

$$1. h = 1. b + 2 1. c - 2 1. a,$$

$$1. h' = 1. 4 + 1. c + 1. d - 1. a.$$

208. *Exponential Equations.* We have explained (209) a method for resolving the equation $a^x = b$, and we have deduced from it the theory of logarithms; but as there are tables actually constructed, nothing prevents the use of them in resolving this sort of equations.

Now, if we take the logarithms of the two members of the equation $a^x = b$, it becomes (216)

$$x \times 1. a = 1. b, \text{ whence } x = \frac{1. b}{1. a}.$$

For example, let us take again the equation $3^x = 15$, which, by the method of (209), has given $x = 2,465$, to within 0,001; we deduce from this equation

$$x = \frac{1. 15}{1. 3} = \frac{1,17609126}{0,47712125} = 2,465 \dots$$

The equation $a^x = b$, is called an *exponential equation of the first order*; but we can have equations of the form

$$a^{b^x} = c; a^{b^{c^x}} = d \dots;$$

we call them *exponential equations of the second, third, ... order.*

In order to form an idea of the expression a^{b^x} , we must suppose that b is raised to a power of a degree indicated by x , and that a is raised to a power of a degree indicated by b^x .

In the same way, $a^{b^{c^x}}$ indicates that after having raised c to the power of a degree indicated by x , we have then raised b to the power of a degree shown by c^x , and finally, that a is raised to a power of a degree indicated by b^{c^x} .

According to these principles, let us take the logarithms of the two members of the equation $a^{b^x} = c$; it becomes

$$b^x \times 1. a = 1. c; \text{ whence } b^x = \frac{1. c}{1. a},$$

or taking again the logarithms

$$x \times 1. b = 1. \frac{c}{a} = 1. 1. c - 1. 1. a; \text{ then } x = \frac{1. 1. c - 1. 1. a}{1. b}.$$

[1. c being a decimal fraction, we can determine its logarithm according to the tables, as we determine the logarithm of any other number.]

Let it be further required to resolve the equation $a^{b^x} = d$.

Taking the logarithms, we have

$$b^{c^x} \times \log a = \log d;$$

whence

$$b^{c^x} = \frac{1. d}{1. a};$$

taking again the logarithms

$$c^x = \frac{1. 1. d - 1. 1. a}{1. b};$$

and performing the same operation upon this equation as upon the preceding,

$$x \times 1. c = 1. \frac{1. 1. d - 1. 1. a}{1. b} = 1. (1. 1. d - 1. 1. a) - 1. 1. b;$$

then

$$x = \frac{1. (1. 1. d - 1. 1. a) - 1. 1. b}{1. c}.$$

We should resolve by a similar process the exponential equations of a higher order. These formulas are exact, considered *algebraically*; but in the applications it is easy to see that they would give values which are only slight approximations, and we could not even form a very just idea of the degree of approximation.

209. *Remark.* It may happen that in the calculation of algebraic expressions, we may be led to take the logarithm of a negative number.

Let it be required, for example, to calculate the expression

$$x = \frac{a^2 - b^2}{c};$$

we have

$$1. x = 1. (a + b) + 1. (a - b) - 1. c,$$

If we suppose $a < b$, $\text{l.}(a - b)$ becomes $\text{l.}(-m)$. How must we interpret this result?

Let us begin by showing that in every system of logarithms, *negative numbers have no logarithms*. Indeed, if the *base* is positive, there exists no power of that base which can produce a negative number; but we cannot take a negative *base*; for that quantity raised to its different powers, would produce numbers, some positive, others negative; now the character of a *base* is, to produce all consecutive numbers, when we raise it to powers which increase progressively.

So that it is with the logarithm of a negative number, as with its square root, fourth root . . . ; $\text{l.}(-m)$ is an absurd or *imaginary* expression.

We should seek in vain to weaken this proposition by the following reasoning;

We know that $(-m)^2 = m^2$; then, by taking the logarithms on both sides,

$$2 \text{l.}(-m) = 2 \text{l.}m; \text{ whence } \text{l.}(-m) = \text{l.}m.$$

This reasoning fails in this point, that we put

$$\text{l.}(-m)^2 = 2 \text{l.}(-m),$$

according to the property of (216); now, this property supposes that we can have the equation $-m = a^x$, a being positive, which is impossible.

Let us return to our object. Two circumstances may be presented.

Either logarithms are employed only as a more simple method of finding the numerical value of the expression; that is, the calculation by logarithms is then a calculation purely *auxiliary*, and its use is not indispensable. Such is the case in which is found the expression

$$x = \frac{a^2 - b^2}{c},$$

which might be calculated directly, without the assistance of logarithms. It is negative, since we have supposed $a < b$; but if we wish absolutely to calculate its value by logarithms, we must commence by changing its sign, which gives

$$\frac{b^2 - a^2}{c}; \text{ whence } \text{l.} \frac{(b^2 - a^2)}{c} = \text{l.}(b + a) + \text{l.}(b - a) - \text{l.}c,$$

an expression which no longer contains any but logarithms of absolute numbers; and when we have obtained the numerical value of $\frac{b^2 - a^2}{c}$, it is sufficient to take the result with the sign —.

Or, the use of logarithms is indispensable in finding the value of an unknown quantity, and then $l. (-m)$ denotes an absurdity or an impossibility in the question.

Let there be, for example, the equation $3^x = -9$.

By applying the logarithms, we find $x \cdot l. 3 = l. -9$, an absurd equation; and indeed, to whatever power we may raise 3, it is impossible to produce — 9.

To resume; whenever in the use of logarithms we have for our object to find more simply the numerical value of a proposed expression, $l. (-m)$ only indicates that the quantities, which enter into the proposed expression, are to undergo one or several changes of sign, before we apply to it the *logarithmic calculation*.

But if, in order to resolve an equation, we are obliged to have recourse to the logarithms, the expression $l. (-m)$ is a symbol of absurdity analogous to the even roots of negative quantities.

210. Proportions and progressions by quotient.

Let there be at first the proportion $a:b::c:x$; we deduce from it $x = \frac{bc}{a}$, whence, by applying the logarithms,

$$l. x = l. b + l. c - l. a,$$

or

$$l. a : l. b : l. c : l. x,$$

which proves that, if four numbers form a proportion, their logarithms form an equidifference, of which the 4th term is the logarithm of the 4th term of the proportion.

Further, let there be a progression by quotient

$$\therefore a : b : c : d : e : f : g : h : \dots$$

It results from the definition (199), that we can write it thus;

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e} = \frac{e}{f} = \frac{f}{g} = \dots,$$

whence taking the logarithms on both sides,

$$l. \frac{a}{b} = l. \frac{b}{c} = l. \frac{c}{d} = l. \frac{d}{e} = l. \frac{e}{f} = \dots,$$

or

$$l. a - l. b = l. b - l. c = l. c - l. d = l. d - l. e = l. e - l. f,$$

or finally,

$$\div 1a.1b.1c.1d.1e....;$$

then, if any numbers $a, b, c, d \dots$ are in a progression by quotients, their logarithms are in a progression by differences. The converse is evident.

This proposition renders the algebraic definition of logarithms (214) similar to that which is given in the Arithmetic; *logarithms are numbers in a progression by difference, corresponding term for term to numbers in a progression by quotient.*

N. B. We have already shown this analogy in our Treatise upon Arithmetic (280).

It is especially in resolving questions relating to progressions by quotients that the use of logarithms is useful.

1. If we call u the last term of a progression by quotient, we have (200)

$$u = a q^{n-1}, \text{ whence } l. u = l. a + (n-1) l. q.$$

For example, let it be proposed to find the 20th term of the progression

$$1 : \frac{3}{2} : \frac{9}{4} : \frac{27}{8} : \dots$$

The formula becomes

$$l. u = l. 1 + 19 (l. 3 - l. 2) = 19 (l. 3 - l. 2), [\text{for } l. 1 = 0];$$

and we obtain, all the calculation being performed,

$$l. u = 3,3457339 = l. 2216,84;$$

whence

$$u = 2216,84 \text{ to within } 0,01.$$

2. If we wish to insert between two given numbers a and b , a number m of mean proportionals, we have, to determine the ratio, (206) the formula

$$q = \sqrt[m+1]{\frac{b}{a}}; \text{ whence } l. q = \frac{l. b - l. a}{m + 1}.$$

Let

$$a = 2, b = 15, m = 50;$$

it becomes

$$l. q = \frac{l. 15 - l. 2}{51},$$

and we obtain, all the calculation being performed,

$$l. q = 0,0171581 = l. 1,040299 ;$$

then

$$q = 1,0402299.$$

If we wish to calculate immediately the twentieth *mean proportional*, which is the twenty-first term of the progression ; we have

$$x = 2 \left(\sqrt[51]{\frac{15}{2}} \right)^{20} ; \text{ whence } \log x = 1.2 + \frac{20 (\log 15 - 1.2)}{51} ;$$

or, all the calculation being performed,

$$l. x = 0,6441913 = l. 4,407489 ;$$

so that the twentieth mean proportional is 4,407489.

(3.) We have found (201), for the expression of the sum of the terms,

$$S = \frac{l. q - a}{q - 1} = \frac{a (q^n - 1)}{q - 1} ;$$

whence

$$l. S = l. a + l. (q^n - 1) - l. (q - 1).$$

We see, according to this formula, that we must commence by calculating the expression q^n , by putting $l. q^n = n l. q$, after which we can easily determine $q^n - 1$, and consequently, $l. (q^n - 1)$. We shall soon have occasion to apply this formula.

(4.) Knowing a , q , and u in the formula $u = a q^{n-1}$, we may wish to find the value of n . Now we have

$$l. u = l. a + (n - 1) l. q ; \text{ whence } n = 1 + \frac{l. u - l. a}{l. q}.$$

Let it be required to find the number of terms of the progression of which the first term is 3, the ratio 2, and the last term 6144. We have

$$n = 1 + \frac{l. 6144 - l. 3}{l. 2} = 1 + \frac{3,31132995}{0,30102999} = 1 + 11 = 12 ;$$

(the quotient $\frac{331132995}{30102999}$ is equal to $11 + \frac{6}{30102999}$; but we neglect the fraction, as it proceeds from the use of logarithms).

211. *Questions relating to Compound Interest.* One of the most important applications of logarithms is that which we make to questions upon the interest of money.

First General Question. *Any sum whatever being put at interest for a certain time, at a given rate, and at compound interest, that is, on the supposition that the interest of each year is added to the principal of the preceding year, it is required what the amount will be at the end of the given time.*

Let us designate by a the sum put at interest, by n the number of years, and by r the interest of one franc per year, which is only the hundredth part of the rate of interest for 100 francs.

Since 1 franc produces r , a sum a will produce ar , at the end of a year; so that at the end of the first year, the principal will have become $a + ar$, or $a(1 + r)$.

Let $a(1 + r) = a'$; this new principal will become, at the end of the second year, $a'(1 + r)$; then the original principal, or a , will have become

$$a'(1 + r), \text{ or } a(1 + r)^2.$$

We should obtain, in the same manner, at the end of the third year, $a(1 + r)^3$, and in general, at the end of the n^{th} year $a(1 + r)^n$. Then expressing this value by A , we shall have the equation

$$A = a(1 + r)^n, \text{ whence } \text{l. } A = \text{l. } a + n \times \text{l. } (1 + r).$$

Application. It is required what a sum of 30,000 francs put at compound interest, at the rate of 5 per cent. will amount to at the end of 30 years.

It is sufficient to make, in the preceding formula,

$$a = 30000, \quad n = 30, \quad r = \frac{5}{100} = 0,05;$$

which gives

$$\begin{array}{r} \text{l. } A = \text{l. } 30000 + 30 \text{ l. } (1,05). \\ \text{l. } 1,5 = 0,021189299; \\ \hline 30 \text{ l. } 1,05 = 0,63567897 \\ \text{l. } 30000 = 4,47712125 \\ \hline \text{l. } A = 5,11280022 = \text{l. } 129658,27. \\ \text{Then} \quad A = 129958,27. \end{array}$$

The formula $A = a(1 + r)^n$ containing four quantities, a , r , n , A , gives the solution of four different problems;

(1.) To determine A , knowing a , r , and n ; this is the question which we have just resolved.

(2.) To determine the sum which must actually be put at interest, in order to obtain, at the end of n years, an amount A , by supposing the principal put at compound interest, at the rate of r for one franc.

Now, from the equation $A = a(1+r)^n$, we deduce

$$l. a = l. A - n l. (1+r);$$

and this new formula will give the value of a .

This second question constitutes the rule of compound discount; for it is reduced to finding the actual value of a sum A payable in n years, by having regard to the interest of the sum and to the interest of the interest.

(3.) To determine the rate of interest at which we must put a sum a , in order to obtain at the end of n years, at compound interest, another sum A .

The formula would be

$$1+r = \sqrt[n]{\frac{A}{a}}, \text{ whence } l. (1+r) = \frac{l. A - l. a}{n}.$$

Knowing $1+r$, we should easily obtain r , and consequently, the rate of interest for 100 francs.

(4.) Finally, to determine the time during which a sum a must be put at compound interest, at the rate of r for 1 franc, in order to produce an amount A .

The formula would be

$$n = \frac{l. A - l. a}{l. (1+r)}.$$

If we wish that A should be double, triple, quadruple . . . of a , the formula would be simplified.

Indeed let $A = ka$; the formula $A = a(1+r)^n$, is reduced to $ka = a(1+r)^n$, whence

$$n = \frac{l. k}{l. (1+r)},$$

that is, the value of n is independent of the principal originally put at interest.

Second General Question. To determine what sum must be actually put at interest in order to receive at the end of each year, a given amount b , so as to be entirely reimbursed for the principal, the interest of the principal, and the interest of the interest after a
Bour. Alg.

number n of years, the interest being at the rate of r for one franc per year.

Let a be the sum required; this principal would become, at the end of n years,

$$a(1+r)^n.$$

It is necessary then in determining what the sums paid each year will become at the end of the n^{th} year, that the amount of the results should be equal to $a(1+r)^n$.

Now, b given at the end of the first year, or at the commencement of the second, becomes at the end of the n^{th} year

$$b(1+r)^{n-1}.$$

In the same way, b given at the end of the second, or at the commencement of the third year, becomes at the end of the n^{th} year

$$b(1+r)^{n-2}.$$

We should find, in a similar manner,

$$b(1+r)^{n-3}, b(1+r)^{n-4}, \dots b(1+r), b,$$

for the values of the other sums b , at the end of the n^{th} year. We have then the equation

$$a(1+r)^n =$$

$$b(1+r)^{n-1} + b(1+r)^{n-2} + b(1+r)^{n-3} + \dots + b(1+r) + b;$$

but the second member of this equation, considered in an inverse order, is evidently the sum of the terms of a progression by quotient, the first term of which is b , the ratio $1+r$, and the number of terms n .

So that this sum has for its expression (201)

$$\frac{b(1+r)^n - b}{1+r-1} \text{ or } \frac{b[(1+r)^n - 1]}{r};$$

then finally we have the equation

$$a(1+r)^n = \frac{b[(1+r)^n - 1]}{r}; \text{ whence } a = \frac{b[(1+r)^n - 1]}{r(1+r)^n},$$

or, applying the logarithms,

$$l.a = l.b + l.[(1+r)^n - 1] - l.r - n.l.(1+r).$$

This new formula, containing four quantities, a , b , r , n , also gives rise to four different problems.

The following are enunciations of several questions which are connected with the preceding.

Required the number of years for which we must put a sum a , at compound interest, at 5 and 10 per cent, in order to double that sum.

(Ans. At 5 per $\frac{5}{100}$, 14 years 2 months; at 10 per $\frac{10}{100}$, 7 years 3 months.)

Required the sum which must be put at interest at the present time, in order to produce for 12 years, at the end of each year, a sum of 1500 francs, so that the whole principal and interest may be repaid at the end of the twelve years, the interest being seven and a half per cent per year.

(Ans. 11602,91.)

A person has bought a property of 100,000 francs, which is to be paid for in fifteen equal payments at compound interest; the rate for each interval of payment is 5 per cent. Required what will be the amount or the quota of each payment.

(Ans. 9634,22.)

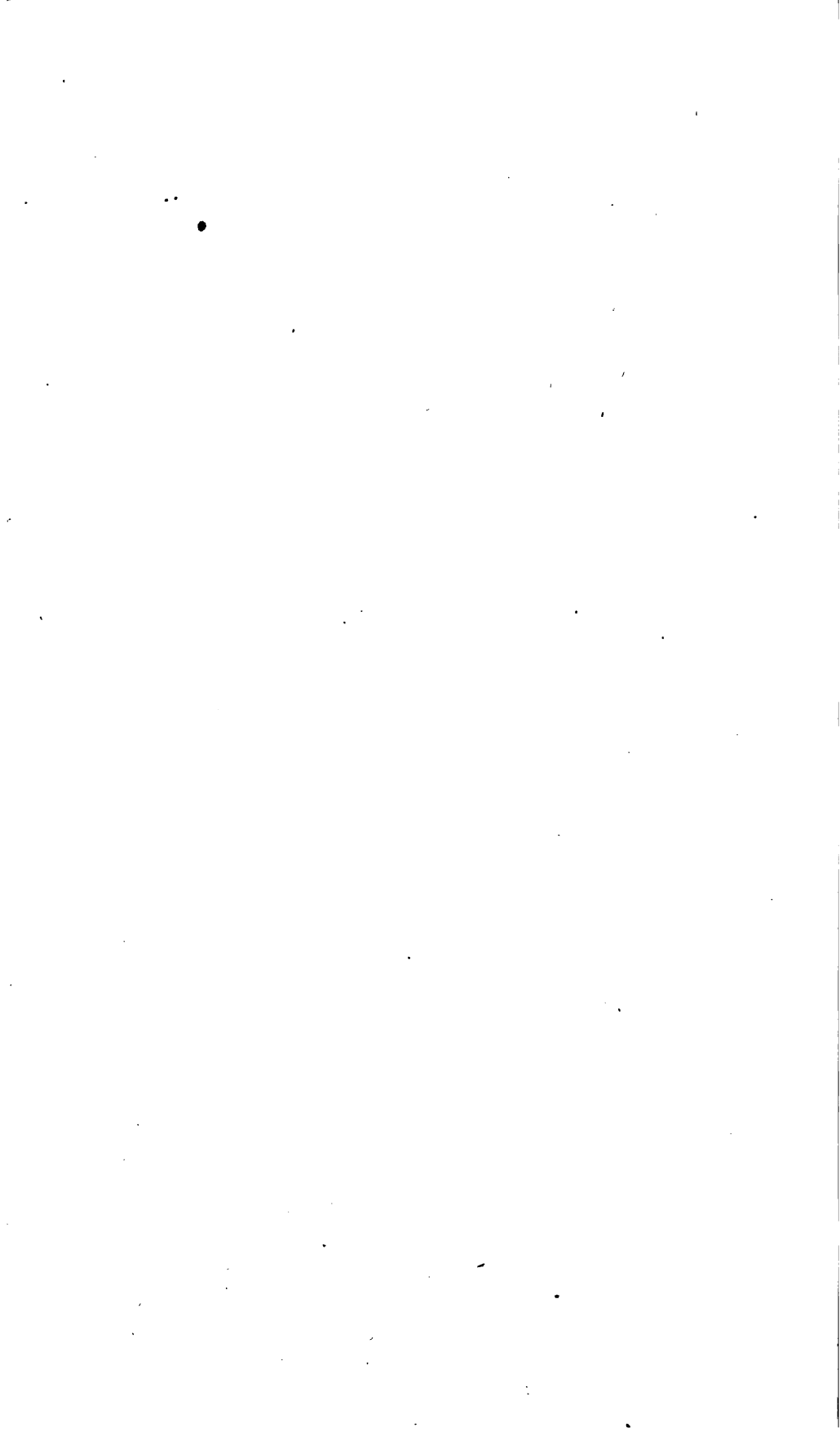
A certain number of men a increases every year by the hundredth part of what it was the preceding year; required in how many years the number will become ten times greater.

(Ans. 231 years nearly.)

Suppose that from a barrel of 100 pints of wine, we draw each day a pint, which we replace by a pint of water; required, 1. how much wine will remain in the barrel, when we have replaced the fiftieth pint; 2. in how many days the wine will be reduced to one half, one third, or one fourth.

Ans. to the first part of the question, $60\frac{1}{2}$ pints.

Ans. to the second part, 69 days for the half, 109 days for the third, and 138 days for the fourth.



NOTE.

IV. *Of Continued Fractions.*

166. CONTINUED FRACTIONS originate in the approximate value of fractions whose terms are considerable, and prime to each other.

In order to be better understood, let there be proposed the fraction $\frac{1}{1 + \frac{1}{2}}$, of which it is easy to show that the two terms are prime to each other, and which for that reason is (157) irreducible.

By leaving this fraction under this form, it becomes difficult to obtain a just idea of it; but if, by means of a known principle, we divide its two terms by 159, which does not change its value, it becomes $(\frac{1}{1 + \frac{1}{2}})$, or by performing the division indicated in the denominator,

$$\frac{1}{3 + \frac{1}{159}}.$$

This being premised, let us neglect for the moment the fraction $\frac{1}{159}$; the fraction $\frac{1}{3}$, which results, is a little greater than the proposed fraction, since we have diminished the denominators.

On the other hand, if, instead of neglecting $\frac{1}{159}$, we replace this fraction by 1, which gives $\frac{1}{3+1}$ or $\frac{1}{4}$, this new fraction is, in its turn, smaller than the proposed fraction, since we have increased the denominator.

Whence we may conclude that the fraction $\frac{1}{1 + \frac{1}{2}}$ is comprehended between $\frac{1}{3}$ and $\frac{1}{4}$. This gives already a sufficiently exact idea of the fraction.

If we wish a greater degree of approximation, we have only to perform the operation with $\frac{1}{159}$, as we have done with $\frac{1}{159}$, and the proposed fraction becomes

$$\frac{1}{3 + \frac{1}{9 + \frac{1}{159}}}.$$

If we neglect $\frac{1}{159}$, $\frac{1}{9}$ is greater than $\frac{1}{159}$; whence it follows that $\frac{1}{3 + \frac{1}{9}}$ is smaller than $\frac{1}{1 + \frac{1}{2}}$. But $\frac{1}{3 + \frac{1}{9}}$ is reduced to $\frac{1}{(\frac{28}{9})}$ or $\frac{9}{28}$;

so that the proposed fraction is still comprehended between $\frac{1}{3}$ and $\frac{2}{3}$.

The difference of these two last fractions, reduced to the same denominator, is

$$\frac{28 - 27}{84} \text{ or } \frac{1}{84}.$$

Then the error we may commit by taking $\frac{1}{3}$ for the value of the proposed fraction is less than $\frac{1}{84}$.

By performing the operation upon $\frac{1}{3}$, as we have done with the preceding, we have

$$\frac{15}{16} = \frac{1}{(\frac{16}{15})} = \frac{1}{1 + \frac{1}{15}};$$

and the proposed fraction may be put under the form

$$\frac{1}{3 + \frac{1}{9 + \frac{1}{1 + \frac{1}{15}}}}$$

Let us neglect $\frac{1}{15}$; the number $\frac{1}{1}$, or 1, is greater than $\frac{1}{15}$, then

$$\frac{1}{9 + \frac{1}{1}}, \text{ or } \frac{1}{10},$$

is smaller than $\frac{1}{9}$.

Then

$$\frac{1}{3 + \frac{1}{9 + \frac{1}{1}}}, \text{ or } \frac{1}{3 + \frac{1}{10}}, \text{ or } \frac{10}{31}, \text{ is greater than } \frac{159}{493}.$$

Whence we see that $\frac{1}{3}$ is comprehended between $\frac{10}{31}$ and $\frac{159}{493}$. The first is too small and the second too large.

Now the difference of these two fractions is $\frac{159}{493} - \frac{10}{31}$, or $\frac{1}{493}$; so that the error which we commit, by taking either $\frac{10}{31}$, or $\frac{159}{493}$, for the value of the proposed fraction, is less than $\frac{1}{493}$.

We see how, by this series of operations, we succeed in finding, in more simple terms, fractions which give the approximate values of another fraction of which the terms are very considerable.

The expression

$$\frac{1}{3 + \frac{1}{9 + \frac{1}{1 + \frac{1}{15}}}}$$

is what we call a continued fraction.

In general, we understand by continued fraction, a fraction which has for its numerator unity, and for its denominator a whole number, plus a fraction which has also unity for its numerator, and for its denominator a whole number, plus a fraction, and so on.

Frequently the proposed fractional number is greater than unity. So that in order to make the definition of a continued fraction more general, we must say : A continued fraction is an expression composed of a whole number, plus a fraction which has for its numerator unity, and for its denominator, &c.

Such is the expression

$$a + \frac{1}{b + \frac{1}{c + \frac{1}{d + \dots}}}$$

($a, b, c, d, \dots$ being whole numbers).

167. By reflecting on the method which has just been pursued in order to reduce $\frac{493}{159}$ to a continued fraction, we see that we first divided 493 by 159, which gave 3 for a quotient, and for a remainder 16. We then divided 159 by 16, which gave for a quotient 9, and for a remainder 15 ; we then divided 16 by 15, which gave 1 for a quotient, and 1 for a remainder. Thence it is easy to determine the following process, in order to reduce a fraction or a fractional number to a continued fraction.

Proceed with the two terms of the proposed fraction, as if to find their greatest common divisor (49). Continue the operation till there is obtained a remainder equal to zero, and the successive quotients at which we shall arrive, will be the determinators of the fractions which constitute the continued fraction.

In the hypothesis in which the proposed number is greater than unity, the first quotient represents the entire portion which enters into the expression of the continued fraction.

We may, according to this method, reduce to continued fractions the two numbers $\frac{65}{2}$ and $\frac{198321}{32212}$.

The following is the form of operations

$$(1.) \quad 149 \left\{ \begin{array}{l} 65 \\ 2 \end{array} \right\} \left\{ \begin{array}{l} 198321 \\ 32212 \end{array} \right\}$$

Then

$$\frac{65}{149} = \frac{1}{2 + \frac{1}{3 + \frac{1}{2 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2}}}}}}$$

(2.)

$$829 \left| \frac{347}{2} \right| \frac{135}{2} \left| \frac{77}{1} \right| \frac{58}{1} \left| \frac{19}{3} \right| \frac{1}{19};$$

whence

$$\frac{829}{347} = 2 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{3 + \frac{1}{19}}}}}}$$

Continued fractions possess a greater number of properties, the discovery of which has been the object of the labours of the most celebrated geometricians. We will here mention the elementary properties only ; those of which we make frequent use, and whose demonstrations are founded on the first principles of Algebra. We refer, for more ample details, to the Additions of Lagrange to Euler's Algebra.

QUESTIONS FOR PRACTICE.

DIVISION.

Cases in which the Divisor is not an aliquot part of the Dividend.

$$1. \frac{a}{1+x} = a - ax + ax^2 - ax^3 + ax^4 - \dots$$

$$2. \frac{a}{1-x} = a + ax + ax^2 + ax^3 + ax^4 + \dots$$

$$3. \frac{a}{x+1} = \frac{a}{x} - \frac{a}{x^2} + \frac{a}{x^3} - \frac{a}{x^4} + \dots$$

$$4. \frac{a}{x-1} = \frac{a}{x} + \frac{a}{x^2} + \frac{a}{x^3} + \frac{a}{x^4} + \dots$$

$$5. \frac{a+x}{b+x} = \frac{a}{b} - \frac{a-b}{b^2}x + \frac{a-b}{b^3}x^2 - \frac{a-b}{b^4}x^3 + \dots$$

$$6. \frac{a-x}{b-x} = \frac{a}{b} + \frac{a-b}{b^2}x + \frac{a-b}{b^3}x^2 + \frac{a-b}{b^4}x^3 + \dots$$

$$7. \frac{x+a}{x-b} = 1 + \frac{a+b}{x} + \frac{b(a+b)}{x^2} + \frac{b^2(a+b)}{x^3} + \dots$$

POWERS OF POWERS.

$$1. [((a^m)^n)^p]^q = a^{mnpq}.$$

$$2. [((a^{-m})^{-n})^p]^q = a^{mnpq}.$$

$$3. [((a^{-m})^{-n})^{-p}]^{-q} = a^{mnpq}.$$

$$4. [((a^m)^{-n})^{-p}]^{-q} = a^{mnpq}.$$

$$5. [(a^3 b c^2)^5]^6 = a^{90} b^{30} c^{60}.$$

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6. $[a^{-2} b^3 c^{-5} f^6 x^{-1}]^{-3} = a^6 b^{-9} c^{15} f^{-18} x^3.$
7. $(a^m b^{-n} c^p d)^r = a^{mr} b^{-nr} c^{pr} d^r.$
8. $(a^{3m-n} f^{2n-1} x^n)^{-3m} = a^{3mn-9mm} f^{3m-6mn} x^{-3mn}.$
9. $[a^3 (a+b)^2]^m = a^{3m} (a+b)^{2m}.$
10. $\left(\frac{a^m b^n c^p d^{-q}}{f^n g^{-m}} \right)^{-r} = \frac{a^{-mr} b^{-nr} c^{-pr} d^{qr}}{f^{-nr} g^{mr}}.$
11. $\left(\frac{a^4 b^5}{c^3 d f} \right)^4 = \frac{a^{16} b^{20}}{c^{12} d^4 f^4}.$
12. $\left[\left(\frac{a^2 b^3}{c d^5} \right)^{-1} \right]^{-2m} = \frac{a^{4m} b^{6m}}{c^{2m} d^{10m}} = \left(\frac{a^4 b^6}{c^2 d^{10}} \right)^m.$
13. $(-a^2)^5 = -a^{10}.$
14. $(-b^{-3})^4 = b^{-12}.$
15. $[(-a)^3]^4 = a^{60}.$
16. $[(-a)^{-3}]^{-5} = -a^{15}.$
17. $[(-a)^{-4}]^{-6} = a^{24}.$
18. $(-a)^{2m} = a^{2m}.$
19. $(-a)^{2m+1} = -a^{2m+1}.$
20. $\left[\left(-\frac{a}{b} \right)^3 \right]^{-4} = \frac{b^{12}}{a^{12}}.$

ROOTS OF LITERAL EXPRESSIONS.

Roots of Simple Literal Expressions.

1. $\sqrt[n]{a^{mn}} = a^n.$
2. $\sqrt[n]{a^{-mn}} = a^{-n}.$
3. $\sqrt[n]{a^{mn} b^{mp} c^{-mq} d^{-mr}} = a^n b^p c^{-q} d^{-r}.$
4. $\sqrt[n]{\frac{a^{mn} b^{mp}}{c^{mq} d^{mr} f^{ns}}} = \frac{a^n b^p}{c^q d^r f^s}.$
5. $\sqrt[3]{9 a^4 b^2 f^{-12} g^{-6n}} = 3 a^{\frac{4}{3}} b^{\frac{2}{3}} f^{-4n} g^{-2n}.$
6. $\sqrt[4]{\frac{1}{2} c^8 x^{2m+4} y^{4p-8n}} = \frac{1}{2} c^2 x^{\frac{m+1}{2}} y^{p-2n}.$

7. $\sqrt[4]{\frac{a^8 b^{20} c^4}{(a+f)^4 h^{12} z^{16}}} = \frac{a^2 b^5 c}{(a+f) h^3 z^4}.$
8. $\sqrt[3]{\frac{27 u^9 x^{12} (a^2 + x^2)^{-6n}}{8 b^{-3n} h^9}} = \frac{3 b^n u^3 x^4}{2 h^3 (a^2 + x^2)^{2n}}.$
9. $\sqrt[5]{\frac{3^{10} a^{-5} (a+b)^5 (2+x)^{-10}}{32 c^5 d^{-15}}} = \frac{9 d^3 (a+b)}{2 a c (2+x)^2}.$
10. $\sqrt[9]{\left(2^{36} a^{45} b^9 \times \frac{(a+b)^{27}}{a^9}\right)} = 2^4 a^4 b (a+b)^3.$
11. $\sqrt{\left(\frac{a^2 b^2 c^2}{d^4} \times \frac{1}{a^6 b^8 f^4}\right)} = \frac{c}{a^2 b^3 d^2 f^2}.$

THE ARITHMETIC OF ROOTS.

Addition and Subtraction.

1. $b \sqrt[m]{a} + c \sqrt[m]{a} - d \sqrt[m]{a} = (b + c - d) \sqrt[m]{a}.$
2. $3 \sqrt[6]{5} + 17 \sqrt[6]{5} - 12 \sqrt[6]{5} - 7 \sqrt[6]{5} = \sqrt[6]{5}.$
3. $6 \sqrt{2} - 5 \sqrt{2} + \frac{3}{4} \sqrt{2} - \frac{1}{2} \sqrt{2} = \frac{5}{4} \sqrt{2}.$
4. $6 \sqrt[4]{\frac{3}{2}} - 2 \sqrt[4]{\frac{3}{2}} + a \sqrt[4]{\frac{3}{2}} - \frac{2b}{c} \sqrt[4]{\frac{3}{2}} = \left(4 + a - \frac{2b}{c}\right) \sqrt[4]{\frac{3}{2}}.$
5. $5 \sqrt[7]{9} - 2 \sqrt[7]{14} + \sqrt[7]{2} - 5 \sqrt[7]{14} - 2 \sqrt[7]{9} = 3 \sqrt[7]{9} - 7 \sqrt[7]{14} + \sqrt[7]{2}.$
6.
$$\begin{array}{l} \text{Added.} \left\{ \begin{array}{l} 10 \sqrt{2} + 5 \sqrt[7]{8} - 7 \sqrt[5]{5} + 2 \sqrt[3]{a} \\ 5 \sqrt{2} + \sqrt[7]{8} + 4 \sqrt[5]{5} - 3 \sqrt[3]{a} \\ -3 \sqrt{2} - 9 \sqrt[7]{8} - 3 \sqrt[5]{5} + \sqrt[3]{a} + \sqrt[5]{a} b \end{array} \right. \\ \hline 12 \sqrt{2} - 3 \sqrt[7]{8} - 6 \sqrt[5]{5} + \sqrt[3]{a} b \end{array}$$
7.
$$\begin{array}{l} \text{Added.} \left\{ \begin{array}{l} 13 \sqrt[5]{12 a^2 b c} + 17 \sqrt[5]{3} - 5 \sqrt[5]{6} \\ 7 \sqrt[5]{12 a^2 b c} + 2 \sqrt[5]{6} + 3 \sqrt[5]{3} - 2 a \sqrt{c} + \frac{7}{2} \sqrt[7]{9 a} \\ -20 \sqrt[5]{12 a^2 b c} + 9 \sqrt[5]{12 a^2 b c} + \sqrt{c} - 3 \sqrt[7]{9 a} \end{array} \right. \\ \hline 20 \sqrt[5]{3} - 3 \sqrt[5]{6} + 9 \sqrt[5]{12 a^2 b c} - (2a-1) \sqrt{c} - \frac{7}{2} \sqrt[7]{9 a}. \end{array}$$

$$8. \begin{array}{r} \text{Subtracted.} \left\{ \begin{array}{l} 18\sqrt{7} - 5\sqrt{6} + 10\sqrt[4]{11} - 3\sqrt[3]{13} \\ 6\sqrt{7} - 2\sqrt{6} + \frac{1}{2}\sqrt[4]{11} + 2\sqrt[3]{13} \end{array} \right. \\ \hline 12\sqrt{7} - 3\sqrt{6} + \frac{3}{2}\sqrt[4]{11} - 5\sqrt[3]{13} \end{array}$$

$$9. \begin{array}{r} \text{Subtracted.} \left\{ \begin{array}{l} 16\sqrt[4]{6}ab - \sqrt[5]{9}c^3 + 3\sqrt[7]{a} - \sqrt{10} \\ -8\sqrt[5]{9}c^3 - 5\sqrt[7]{a} + 3\sqrt[4]{6}ab + 2\sqrt[4]{10} \end{array} \right. \\ \hline 13\sqrt[4]{6}ab + 7\sqrt[5]{9}c^3 + 8\sqrt[7]{a} - \sqrt{10} - 2\sqrt[4]{10}. \end{array}$$

Division.

$$1. \sqrt[m]{a} : \sqrt[m]{b} = \sqrt[m]{\frac{a}{b}}.$$

$$2. c\sqrt[m]{a} : d\sqrt[m]{b} = \frac{c}{d}\sqrt[m]{\frac{a}{b}}.$$

$$3. a : \sqrt[m]{b} = \sqrt[m]{\frac{a^m}{b}}.$$

$$4. a : \sqrt{a} = \sqrt{a}.$$

$$5. 2a^2b^3c^3 : 4\sqrt[3]{a^3b}c^5d = \frac{1}{2}\sqrt[3]{\frac{b^5c^4}{d}}.$$

$$6. \sqrt[5]{a}b^{n-1}c^2 : \sqrt[5]{\frac{a^3b^3}{dc^{n-1}}} = \sqrt[5]{\frac{b^{n-3}c^{n+1}d}{a^2}}.$$

$$7. \sqrt[n]{\frac{f^3g^2}{dx^3}} : \sqrt[n]{\frac{fg}{dx}} = \sqrt[n]{\frac{f^2g}{x^4}}.$$

$$8. \sqrt[3]{a^2bc} : \sqrt[5]{ab^3c^3} = \sqrt[15]{\frac{a^7}{b^4c^4}}.$$

$$9. \sqrt[4]{\frac{a}{b}} : \sqrt{\frac{a}{b}} = \sqrt[4]{\frac{b}{a}}.$$

$$10. 4\sqrt[3]{12} : 2\sqrt{3} = 2\sqrt[6]{\frac{1}{3}}.$$

$$11. \sqrt[5]{64} : 2 = \sqrt[5]{2}.$$

12. $\sqrt[2n]{\frac{a^m b}{c^3 d}} : \sqrt[3n]{\frac{a^{m-1} c^3}{d^5}} = \sqrt[6n]{\frac{a^{m+2} b^3 d^7}{c^{12}}}$.
13. $c \sqrt{(a^2 - x^2)} : \sqrt{(a + x)} = c \sqrt{(a - x)}$.
14. $\sqrt{(a^2 b - b^3 c)} : \sqrt{(a - c)} = b$.
15. $\sqrt{(a^2 - z^2)} : (a - z) \sqrt{\frac{a + z}{a - z}}$.

EXPRESSIONS.

1. $\sqrt[m]{a^n} = a^{\frac{n}{m}}$.
2. $\frac{1}{\sqrt[m]{a^n}} = a^{-\frac{n}{m}}$.
3. $\sqrt[m]{a^n b^p c^q} = a^{\frac{n}{m}} b^{\frac{p}{m}} c^{\frac{q}{m}} = (a^n b^p c^q)^{\frac{1}{m}}$.
4. $\sqrt[m]{\frac{a^n b^p}{c^r d^s e^t}} = a^{\frac{n}{m}} b^{\frac{p}{m}} c^{-\frac{r}{m}} d^{-\frac{s}{m}} e^{-\frac{t}{m}}$.
5. $c \sqrt[4]{a^3} + \frac{d}{\sqrt{a^2}} = c a^{\frac{3}{4}} + d a^{-\frac{1}{2}}$.
6. $\sqrt[5]{a^2 b c} = a^{\frac{2}{5}} b^{\frac{1}{5}} c^{\frac{1}{5}} = (a^2 b c)^{\frac{1}{5}}$.
7. $\sqrt[6]{\frac{a^5 b^7}{c^{12}}} + \sqrt[8]{\frac{a^6 b^4}{d^{20}}} = a^{\frac{5}{8}} b^{\frac{7}{8}} c^{-\frac{2}{3}} + a^{\frac{3}{4}} b^{\frac{1}{2}} d^{-\frac{5}{4}}$.
8. $\frac{\sqrt[3]{(c + d)}}{\sqrt{c^5}} = (c + d)^{\frac{1}{3}} c^{-\frac{5}{6}}$.
9. $\frac{\sqrt{(a^2 - x^2)}}{\sqrt{a} \cdot \sqrt{(a + x)}} = a^{-\frac{1}{2}} (a + x)^{-\frac{1}{2}} (a^2 - x^2)^{\frac{1}{2}}$.
10. $\frac{\sqrt[3]{(a + b)^7 c^4}}{\sqrt[5]{f^2} \cdot \sqrt[6]{g^2}} = \frac{c^{\frac{4}{3}} (a + b)^{\frac{7}{3}}}{f^{\frac{2}{5}} g^{\frac{2}{3}}} = c^{\frac{4}{3}} f^{-\frac{2}{5}} g^{-\frac{1}{3}} (a + b)^{\frac{7}{3}}$.

ARITHMETIC OF FRACTIONAL POWERS.

Multiplication.

$$1. a^{\frac{m}{n}} \times a^{\frac{p}{q}} = a^{\frac{m}{n} + \frac{p}{q}} = a^{\frac{mq+np}{nq}}.$$

$$2. a^{\frac{m}{n}} \times a^{-\frac{p}{q}} = a^{\frac{m}{n} - \frac{p}{q}} = a^{\frac{mq-np}{nq}}.$$

$$3. a^{-\frac{m}{n}} \times a^{-\frac{p}{q}} = a^{-\frac{m}{n} - \frac{p}{q}} = a^{-\left(\frac{m}{n} + \frac{p}{q}\right)} = a^{-\frac{mq+np}{nq}}.$$

$$4. a^{\frac{3}{4}} \times a^{\frac{5}{3}} = a^{\frac{3 \cdot 3}{12}} = a^2 \sqrt[12]{a^5}.$$

$$5. a^{-\frac{1}{2}} \times a^{\frac{7}{4}} \times a^{-\frac{1}{8}} = a^{\frac{2 \cdot 0}{8}} = a^2 \sqrt[8]{a}.$$

$$6. a^{-\frac{3}{4}} \times a^{-\frac{7}{8}} = a^{-\frac{1 \cdot 3}{8}} = \frac{1}{a \sqrt[8]{a^3}}.$$

$$7. a^{-\frac{3}{4}} b^{-2} \times a^{\frac{5}{6}} b^{\frac{1}{2}} c = a^{\frac{1 \cdot 2}{12}} b^{-\frac{3}{2}} c = \frac{c}{b} \sqrt[12]{\frac{a}{b^6}}.$$

$$8. \frac{a}{b^{\frac{1}{2}} c^{\frac{3}{4}}} \times \frac{a^{\frac{7}{8}} b}{c^{-\frac{1}{2}}} = a^{\frac{1 \cdot 5}{8}} b^{\frac{1}{2}} c^{-\frac{1}{4}} = a^2 \sqrt[8]{\frac{b^4}{a c^2}}.$$

$$9. \sqrt[5]{a^{12}} \times \sqrt[7]{a^3} \times \sqrt[6]{a^4} = a^{\frac{1 \cdot 2}{5}} \cdot a^{\frac{3}{7}} \cdot a^{\frac{2}{3}} = a^{\frac{3 \cdot 6 \cdot 7}{105}} = a^3 \sqrt[105]{a^{52}}.$$

$$10. \sqrt[5]{\sqrt[3]{a^{24}}} \times \sqrt[6]{\sqrt[4]{a^9}} = a^{\frac{2 \cdot 2}{15}} \cdot a^{\frac{3}{8}} = \sqrt[120]{a^{61}} = \sqrt[120]{a} \cdot \sqrt[120]{a}.$$

$$11. \sqrt[5]{\frac{(c^2 - y^2)^3}{(a+x)^8}} \times \sqrt[6]{\frac{(c^2 - y^2)^{\frac{3}{2}}}{a+x}} = (c^2 - y^2)^{\frac{1 \cdot 7}{10}} (a+x)^{-\frac{4 \cdot 3}{10}} \\ = \frac{c^2 - y^2}{(a+x)^3} \sqrt[60]{\frac{(a+x)^{14}}{(c^2 - y^2)^9}}.$$

Division.

$$1. a^{\frac{m}{n}} : a^{\frac{p}{q}} = a^{\frac{m}{n} - \frac{p}{q}} = a^{\frac{mq-np}{nq}}.$$

$$2. a^{\frac{m}{n}} : a^{-\frac{p}{q}} = a^{\frac{m}{n} + \frac{p}{q}} = a^{\frac{mq+np}{nq}}.$$

$$3. a^{-\frac{m}{n}} : a^{\frac{p}{q}} = a^{-\frac{m}{n} - \frac{p}{q}} = a^{-\left(\frac{m}{n} + \frac{p}{q}\right)} = a^{-\frac{mq+np}{nq}}.$$

$$4. a^{\frac{m}{n}} : a^{\frac{p}{q}} = a^{\frac{p}{q} - \frac{m}{n}} = a^{\frac{np - mq}{nq}} = a^{\frac{mq - np}{nq}}.$$

$$5. c a^{\frac{3}{4}} : d a^{\frac{5}{8}} = \frac{c a^{-\frac{1}{8}}}{d} = \frac{c}{d \sqrt[8]{a}}.$$

$$6. a^{\frac{3}{5}} b^{\frac{1}{2}} : a^{-\frac{7}{5}} b^{-\frac{1}{4}} c = \frac{a^2 b^{\frac{3}{4}}}{c} = \frac{a^2}{c} \sqrt[4]{b^3}.$$

$$7. h : \frac{a^{\frac{1}{3}} b^{\frac{1}{4}}}{c d^{\frac{1}{2}}} = \frac{c h \sqrt{d}}{\sqrt[12]{a^4 b^3}}.$$

$$8. \frac{a^{-\frac{2}{3}} b^{\frac{2}{3}}}{c^{\frac{1}{3}} d^3} : \frac{a^{-\frac{2}{3}} d^{\frac{1}{3}}}{b^{\frac{2}{3}} c} = \frac{a^{\frac{1}{3}} b^{\frac{3}{5}} c}{c^{\frac{1}{3}} d^{\frac{2}{3}}} = \frac{a^3 b^3 c^{60} b^{16} d^{90}}{d^7 \sqrt{a^{15} c^{10}}} \\ = \frac{a^3 b^3 c^{19}}{d^7 \sqrt[19]{a^3 c^3}} \sqrt[15]{b^4}.$$

$$9. (a^3 - 2\sqrt[4]{a^2 b^3} - a^3 \sqrt[6]{a^3 b^2} + 2b \sqrt[12]{b}) : (\sqrt{a} - \sqrt[3]{b}) \\ = (a^3 - 2a^{\frac{1}{3}} b^{\frac{3}{4}} - a^{\frac{5}{3}} b^{\frac{1}{3}} + 2b^{\frac{1}{3}}) : (a^{\frac{1}{3}} - b^{\frac{1}{3}}) \\ = a^{\frac{3}{2}} - 2b^{\frac{2}{3}} = a^2 \sqrt{a} - 2\sqrt[3]{b^2}.$$

APPLICATION TO THE FINDING LOGARITHMS OF PRODUCTS, QUOTIENTS, POWERS, AND ROOTS.

For General or Literal Expressions.

$$1. \log a b c d = \log a + \log b + \log c + \log d.$$

$$2. \log \frac{fg}{cd} = \log f + \log g - \log c - \log d.$$

$$3. \log a^m b^n c^p = m \log a + n \log b + p \log c.$$

$$4. \log \frac{a^m b^{-n}}{c^p d^q} = m \log a - n \log b - p \log c - q \log d.$$

$$5. \log a^{\frac{m}{n}} b^{-\frac{p}{q}} c = \frac{m}{n} \log a - \frac{p}{q} \log b + \log c.$$

$$6. \log \sqrt[n]{a^m b^{-n} c^{\frac{p}{q}}} = \frac{m}{n} \log a - \log b + \frac{p}{nq} \log c.$$

7. $\log \frac{a \sqrt[n]{c^m}}{b \sqrt{d}} = \log a + \frac{m}{n} \log c - \log b - \frac{1}{2} \log d.$
8. $\log \frac{(a+b)^n c^m}{(c+d) \sqrt{d^3}} = n \log(a+b) + m \log c - \log(c+d) - \frac{3}{2} \log d.$
9. $\log \frac{1}{(a+b)^m} = -m \log(a+b).$
10. $\log \frac{1}{\sqrt[n]{a+b}} = -\frac{1}{n} \log(a+b).$
11. $\log \sqrt[m]{(a^2-x^2)} = \frac{1}{m} \log(a^2-x^2) = \frac{1}{m} \log(a+x)$
 $\quad \quad \quad + \frac{1}{m} \log(a-x).$
12. $x \log a = \log a^x.$
13. $n \log a + m \log b - p \log c = \log \frac{a^n b^m}{c^p}.$
14. $n \log(a+y) + \log c - m \log(a-y) = \log \frac{c(a+y)^n}{(a-y)^m}.$
15. $\frac{1}{n} \log(2a+3b) - \frac{2}{3} \log c = \log \frac{\sqrt[n]{(2a+3b)}}{\sqrt[3]{c^2}}.$

THE END.



